Continuous-time Sequential Decision Feedback: Revisited

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Abstract — **Sequential feedback communications has wide ranging applications such as low power communications, error-resilience protocols etc. Two kinds of feedback communication systems can be indentified: information feedback and decision feedback. Continuous-time sequential decision feedback communication is the focus of this paper. The case when the detector test statistic is a Poisson random walk process with reversals whose velocities are governed by a Markov jump process is addressed. Some mathematical properties of this feedback system are investigated.**

I. Introduction

In general, there are two classes of signal detection procedures using hypothesis testing — fixed sample size and sequential test. Example of a fixed sample size test is the Neyman-Pearson binary hypothesis test [1] that utilizes a fixed number of observations to minimize the probability of miss (α_{10}=P(decide $H_0|H_1$ true)) for a given false alarm (α_{01}=P(decide $H_1|H_0$ true)) constraint. Here, H_0 and H_1 denote null and alternative hypothesis, respectively. Usually, in fixed sample size detection, both types of error probabilities cannot be controlled simultaneously [1]. However, in many applications the error probabilities are critical but the total number of observations may not be a constraint. Examples include communication systems with feedback, serial acquisition of direct-sequence spread spectrum signals, pattern recognition, and automatic modulation classification. In a communication system employing a feedback channel the state of the received signal is often fed back to the transmitter ([19] and [20]). Based on the feedback the transmitter either chooses to retransmit the same information or sends new information. This feedback can be one of two types [19]:

- *Information feedback* — receiver relays all or part of the received information to the transmitter and any errors are corrected at the transmitter
- *Decision feedback* — receiver requests retransmission of only uncertain data

Performance analysis of these two types of systems is important due to the fact that many state-of-the-art communication systems employ a feedback channel for rate control, power control etc. We aim to analyze a sequential decision feedback system in this paper. The scenario we consider is similar to the ones presented in [19] and [20] but only in a more general setting.

It is known that Wald's sequential probability ratio test (SPRT) [2] based binary sequential signal detection for independent, identically distributed (i.i.d.) observations is optimum—minimizes the expected value of the required sample size (also called the average sample number (ASN)) among all tests for which the error probabilities have a predefined constraint. The problem of sequential hypothesis testing when the observations are independent and identically distributed (i.i.d.) has been widely studied [2]-[6]; however, in many applications the observations are not independent [7]. For correlated observations the optimum sequential test is shown to be in the form of a generalized sequential probability ratio test (GSPRT) [8]. Sequential detection of signals in autoregressive noise is studied in [9]. Here, the SPRT and a sequential linear detector are compared in terms of the ASN for fixed error probabilities. It is shown that both these detectors have the same asymptotic relative efficiency. In [10], quantization and conditional testing for sequential detection is introduced. Wald's identity has been generalized for dependent observations in [11]. In [12], a non-parametric SPRT detecting signals in additive Markov noise is analyzed. Other works that deal with the non-i.i.d. case can be found in [13]-[17].

In [19], the effect of sequential decision feedback on communication over the Gaussian channel is studied. Coherent communication over a forward channel perturbed by additive white Gaussian noise and a noiseless feedback channel is considered that employs a sequential decision feedback strategy. Analysis of this systems is based on the sequential analysis introduced by Wald [2]. For a binary uncoded communication over the Gaussian channel an ideal feedback channel is observed to give a savings of approximately a factor of four in the energy-to-noise ratio relative to the conventional one-way system for the same error probability. Clearly, this is very useful for low power communications. A similar analysis is used by Kramer [20] to study the performance of another sequential decision feedback communication system. Exact expression for the error probability is derived and the systems is shown to perform better than [19]. The error probability is observed to be always less than the method in [19] and shows rapid improvement for rates near channel capacity.

The aim of this paper is to revisit sequential decision feedback communication for binary signaling. The effect of sequential decision feedback is presented when the detector test statistic is a Poisson random walk process with reversals and the velocity of this process is controlled by a Markov jump process. The paper is organized as follows. The sequential communication problem is introduced in Section II, Section III contains analysis of a sequential detector and brief conclusions are given in Section IV.

II. Problem Definition

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $t \in [0, \infty)$, be a stochastic basis with standard assumptions about the monotonicity and right-continuity of the σ-algebras $\mathcal{F}_t$. Let $\mathcal{F}_t = \sigma\{X(u), 0 \leq u \leq t\}$. A sequential decision rule is the pair (τ, d) where τ is a Markov stopping time with respect to the family $\{\mathcal{F}_t\}$ (*i.e.*,

$\{\tau \leq t\} \in \mathcal{F}_t$), and d, the terminal decision function is $\mathcal{F}_\tau$ measurable. $X = \{X(t),\ t \geq 0\}$ is interpreted as describing the position of a random walk process. Let the velocity process $Y = \{Y(t),\ t \geq 0\}$ of this random walk be governed by two Poisson processes. Suppose the random walk moves with a constant speed V along the real-line with the sign of the velocity changing (*i.e.*, change of direction) according to the following probability law,

$$
\begin{aligned}
P(\text{turn in } (t, t+dt)|\text{moving forward}) &= \mu_1 dt + o(dt)\\
P(\text{turn in } (t, t+dt)|\text{moving backward}) &= \mu_2 dt + o(dt)
\end{aligned} \tag{1}
$$

Then this is a kind of Poisson process with reversals. The rates of the Poisson process are μ_1 and μ_2.

If $\{X(t)\}$ is the process observed by a sequential detector then the sequential detection problem under consideration is then defined as a test between the hypotheses,

$$
\begin{aligned}
H_0 &: \mu = (\mu_1^0, \mu_2^0)\\
H_1 &: \mu = (\mu_1^1, \mu_2^1).
\end{aligned} \tag{2}
$$

Therefore, the random walk process is driven by velocity process $Y(t)$ that is a Markov jump process with generator matrix

$$
\begin{pmatrix} -\mu_2 & \mu_2 \\ \mu_1 & -\mu_1 \end{pmatrix} \tag{3}
$$

The suffix has been dropped for notational convenience. Without loss of generality we assume that $Y(0) = -V$ or V with equal probability. Let $-B$ and A, $B, A > 0$ denote two constant boundaries. Then, the sequential test accepts H_0 or H_1 according to whether $X(t)$ is absorbed in $-B$ or A respectively and $X(t)$ is observed until a decision is made. If $p(x_0, t, x)$ denotes the probability density of $X(t)$ when $X(0) = x_0$ then it can be shown that,

$$
\frac{\partial^2 p}{\partial t^2} + 2\bar{\mu}\frac{\partial p}{\partial t} = V^2 \frac{\partial^2 p}{\partial x_0^2} + 2\bar{\mu} u \frac{\partial p}{\partial x_0} \tag{4}
$$

which is a diffusion equation with drift and, $\bar{\mu} = \frac{\mu_1+\mu_2}{2}$ and $u = \frac{V(\mu_2-\mu_1)}{2\bar{\mu}}$. We note that Eq. (4) is a generalization of the telegrapher's equation [21]. When $\mu_1 = \mu_2 = \mu$, Eq. (4) reduces to the telegrapher's equation, namely,

$$
\frac{\partial^2 p}{\partial t^2} + 2\mu\frac{\partial p}{\partial t} = V^2 \frac{\partial^2 p}{\partial x_0^2} \tag{5}
$$

because $\bar{\mu} = \mu$ and $u = 0$. Clearly, this represents a diffusion without drift. If $\bar{\mu}$ is large such that the diffusion coefficient $\frac{V^2}{\bar{\mu}} \to \sigma^2$ then using Eq. (4) we obtain

$$
\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x_0^2} + u\frac{\partial p}{\partial x_0} = \frac{\partial p}{\partial t} \tag{6}
$$

which is the backward diffusion equation for the Wiener process [18]. Solving for $p(.,.,.)$ by using some linear transformations the required pdf is given by

$$
\begin{aligned}
p(x_0, t, x) &= \frac{1}{2}\left[e^{-\mu_1 t}\delta(x - x_0 - Vt) + e^{-\mu_2 t}\delta(x - x_0 + Vt)\right] +\\
&\quad \frac{\bar{\mu}}{V} e^{-\bar{\mu}\left[t - \left(\frac{u}{V^2}\right)(x - x_0)\right]} \times\\
&\quad \left[I_0(y) + \bar{\mu}\left(t - \left(\frac{u}{V^2}\right)(x - x_0)\right)\frac{I_1(y)}{y}\right] \times\\
&\quad H(Vt - |x - x_0|)
\end{aligned} \tag{7}
$$

where $H(.)$ is the Heaviside function defined as

$$
H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0, \end{cases} \tag{8}
$$

$y = \frac{\bar{\mu}}{V}\sqrt{V^2\left[t - \left(\frac{u}{V^2}\right)(x - x_0)\right]^2 - (x - x_0 - ut)^2}$, $I_0(.)$ and $I_1(.)$ denote the modified Bessel function of the first kind of order zero and one respectively.

III. Sequential Detector

In this section we analyze the sequential detector. We note that the sequential detector is described by the random walk restricted by the two absorbing barriers at $-B$ and A. Absorption at $-B$ corresponds to deciding H_0 and absorption at A corresponds to H_1. We prove that this sequential test terminates w.p. 1.

Theorem 1 *The sequential test under consideration terminates with probability 1,* i.e., $P(\tau < \infty) = 1$.

Proof. Without loss of generality let us assume $x_0 = 0$. Then

$$
y = \frac{\bar{\mu}}{V^2}\sqrt{[V^2 - u^2][V^2 t^2 - x^2]}.
$$

The test does not terminate if $X(t) \in (-B, A)$. The probability of this event is given by

$$
\begin{aligned}
P(X(t) \in (-B, A)) &= \int_{-B}^{A} p(0, t, x)dx\\
&= \frac{\bar{\mu}}{V}e^{-\bar{\mu}t}\int_{-B}^{A} I_0(y)dx +\\
&\quad \frac{\bar{\mu}^2}{V}e^{-\bar{\mu}t}\int_{-B}^{A}\left[t - \left(\frac{u}{V^2}\right)x\right]\frac{I_1(y)}{y}dx
\end{aligned}
$$

Since $I_0(.)$ and $I_1(.)$ are increasing functions we can write

$$
\begin{aligned}
I_0\left(\frac{\bar{\mu}}{V^2}\sqrt{[V^2 - u^2][V^2 t^2 - x^2]}\right) &\leq I_0\left(\frac{\bar{\mu}}{V^2}\sqrt{[V^2 - u^2]V^2 t^2}\right)\\
&= I_0\left(\frac{\bar{\mu}t}{V}\sqrt{V^2 - u^2}\right)
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
&\frac{I_1\left(\frac{\bar{\mu}}{V^2}\sqrt{[V^2 - u^2][V^2 t^2 - x^2]}\right)}{\left(\frac{\bar{\mu}}{V^2}\sqrt{[V^2 - u^2][V^2 t^2 - x^2]}\right)}\\
&\leq \frac{I_1\left(\frac{\bar{\mu}t}{V}\sqrt{V^2 - u^2}\right)}{\left(\frac{u}{V^2}\right)\sqrt{[V^2 - u^2][V^2 t^2 - \max^2(A, |B|)]}}
\end{aligned} \tag{10}
$$

Therefore,

$$
\begin{aligned}
P(X(t) \in (-B, A)) &\leq \frac{\bar{\mu}}{V}e^{-\bar{\mu}t} I_0\left(\frac{\bar{\mu}t}{V}\sqrt{V^2 - u^2}\right)(A + B) +\\
&\quad \frac{\bar{\mu}Ve^{-\bar{\mu}t} I_1\left(\frac{\bar{\mu}t}{V}\sqrt{V^2 - u^2}\right)}{\sqrt{[V^2 - u^2][V^2 t^2 - \max^2(A, |B|)]}} \times\\
&\quad \left[t(A + B) - \left(\frac{u}{V^2}\right)\left(\frac{A^2 - B^2}{2}\right)\right]
\end{aligned} \tag{11}
$$

Using the following asymptotic result from [25] for a mth order modified Bessel function,

$$
I_m(t) \sim \sqrt{\frac{1}{2\pi t}}e^t, \quad \text{as } t \to \infty
$$

in Eq. (11), for sufficiently large t and observing that $\frac{\sqrt{V^2-u^2}}{V} \leq 1$ we see that the right hand side of the above equation tends to zero exponentially as $t \to \infty$. Therefore, it follows that $E(\tau) < \infty$.

Next, we use the method of Fourier series to find the absorption pdf of the process denoted by $\hat{p}(x_0, t, x)$. It is seen that the boundary and the initial condition the pdf should satisfy are

$$\hat{p}(-B, t, x) = 0; \quad \hat{p}(A, t, x) = 0 \tag{12}$$

$$\hat{p}(x_0, 0, x) = \delta(x - x_0) \tag{13}$$

Let

$$\hat{p}(x_0, t, x) = e^{k_1 x_0 - k_2 t} sin\left(\frac{n\pi(x_0 + B)}{A + B}\right), \quad n, 1, 2, \dots \tag{14}$$

be the form of the solution to the diffusion equation in Eq. (4). Clearly, this solution satisfies boundary conditions in Eq. (12). The constants k_1 and k_2 will be evaluated such that the initial condition is satisfied. From Eq. (14) we obtain the following

$$\begin{aligned}
\frac{\partial \hat{p}}{\partial t} &= -k_2 e^{k_1 x_0 - k_2 t} sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) \\
\frac{\partial^2 \hat{p}}{\partial t^2} &= k_2^2 e^{k_1 x_0 - k_2 t} sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) \\
\frac{\partial \hat{p}}{\partial x_0} &= k_1 e^{x_0 - k_2 t} sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) + \\
& \quad e^{k_1 x_0 - k_2 t}\left(\frac{n\pi}{A + B}\right) cos\left(\frac{n\pi(x_0 + B)}{A + B}\right) \\
\frac{\partial^2 \hat{p}}{\partial x_0^2} &= k_1^2 e^{k_1 x_0 - k_2 t} sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) + \\
& \quad 2k_1 e^{k_1 x_0 - k_2 t}\left(\frac{n\pi}{A + B}\right) cos\left(\frac{n\pi(x_0 + B)}{A + B}\right) - \\
& \quad \left(\frac{n\pi}{A + B}\right)^2 sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) e^{k x_0 - k_2 t}
\end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (4) and performing algebraic manipulations, we get

$$\begin{aligned}
\hat{p}(x_0, t, x) &= \frac{2}{A + B}\sum_{n=1}^{\infty} e^{\frac{\bar{\mu}u}{V^2}(x - x_0) - \bar{\mu}t} \times \\
& \quad sin\left(\frac{n\pi(x + B)}{A + B}\right) sin\left(\frac{n\pi(x_0 + B)}{A + B}\right) \times \\
& \quad cos\left[\sqrt{\left(\frac{n\pi V}{A + B}\right)^2 - \bar{\mu}^2\left(1 - \left(\frac{u}{V}\right)^2\right)}\right] t
\end{aligned} \tag{16}$$

This is the pdf of the random walk process with two absorbing barriers.

A Moments of the First Exit Time

We know that $E(\tau) < \infty$. Let $\tau = \inf\{t > 0; X(t) \notin (-B, A) | X(0) = x_0, -B < x_0 < A\}$ denote the first exit time of the stochastic process $X(t)$ from the interval $(-B, A)$. Proceeding from the backward equation we obtain,

$$\frac{\partial^2 \hat{p}}{\partial t^2} + 2\bar{\mu}\frac{\partial \hat{p}}{\partial t} = V^2 \frac{\partial^2 \hat{p}}{\partial x_0^2} + 2\bar{\mu}u\frac{\partial \hat{p}}{\partial x_0} \tag{17}$$

If

$$1 - G(t|x_0) = p(\tau > t) = \int_{-B}^{A} \hat{p}(x_0, t, x)dx \tag{18}$$

then the pdf of τ is $g(t|x_0) = \frac{dG}{dt}$ from which we see that $g(t|x_0) = -\frac{\partial p(\tau > t)}{\partial t}$. Let the moment generating function (mgf) be

$$\Upsilon(s|x_0) = \int_0^{\infty} e^{-st} g(t|x_0)dt \tag{19}$$

If $P_1(x_0, t) = \int_{-B}^{A} p(x_0, t, x)dx$ then we see that it satisfies Eq. (17), *i.e.*,

$$\frac{\partial^2 P_1}{\partial t^2} + 2\bar{\mu}\frac{\partial P_1}{\partial t} = V^2 \frac{\partial^2 P_1}{\partial x_0^2} + 2\bar{\mu}u\frac{\partial P_1}{\partial x_0} \tag{20}$$

Taking the partial derivative with respect to t on both sides and changing the order of differentiation shows that

$$\frac{\partial^2 g}{\partial t^2} + 2\bar{\mu}\frac{\partial g}{\partial t} = V^2 \frac{\partial^2 g}{\partial x_0^2} + 2\bar{\mu}u\frac{\partial g}{\partial x_0} \tag{21}$$

By taking the Laplace transform

$$V^2 \frac{d^2 \Upsilon}{dx_0^2} + 2\bar{\mu}u\frac{d\Upsilon}{dx_0} = s^2\Upsilon + 2\bar{\mu}s\Upsilon \tag{22}$$

with $-B < x_0 < A$ and boundary conditions, $\Upsilon(-B) = 1$ and $\Upsilon(A) = 1$. Since $\Upsilon(x_0)$ is the moment generating function of the first exit time τ we have $\Upsilon(x_0) = \sum_{n=0}^{\infty}(-s)^n \frac{m_n(x_0)}{n!}$, where $m_n(x_0) = E(\tau^n)$. Substituting this into Eq. (22) we have,

$$\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left[V^2 \frac{d^2 m_n}{dx_0^2} + 2\bar{\mu}u\frac{dm}{dx_0} - (s^2 + 2m\bar{\mu}us\right] = 0 \tag{23}$$

Equating the co-efficients of s^n to zero,

$$\begin{aligned}
& V^2 \frac{(-1)^n}{n!}\frac{d^2 m_n}{dx_0^2} + 2\bar{\mu}u\frac{(-1)^n}{n!}\frac{dm}{dx_0} \\
&= (-1)^{n-2}\frac{m_{n-2}}{(n-2)!} + 2\bar{\mu}(-1)^{n-1}\frac{m_{n-1}}{(n-1)!}
\end{aligned}$$

which implies

$$V^2 \frac{d^2 m_n}{dx_0^2} + 2\bar{\mu}u\frac{dm_n}{dx_0} = n(n-1)m_{n-2} - 2\bar{\mu}nm_{n-1} \tag{24}$$

Therefore, for $n = 1$, $m_1 = E(\tau)$ satisfies the following differential equation

$$\frac{d^2 m_1}{dx_0^2} + \frac{2\bar{\mu}u}{V^2}\frac{dm_1}{dx_0} = -\frac{-2\bar{\mu}}{V^2} \tag{25}$$

with boundary conditions $m_1(-B) = m_1(A) = 0$. It is easy to see that the complementary solution to this differential equation is of the form

$$m_1^{(c)}(x_0) = c_1 + c_2 e^{-\left(\frac{2\bar{\mu}u}{V^2}\right)x_0} \tag{26}$$

for two arbitrary constants c_1 and c_2. Assuming a particular solution of the form $m_1^{(p)}(x_0) = c_3 x_0$ for some constant c_3 and substituting in Eq. (25) we see that $c_3 = -\frac{1}{u}$. Therefore, the complete solution to Eq. (25) is given by $m_1(x_0) = m_1^{(c)}(x_0) + m_1^{(p)}(x_0)$. Now, using the boundary conditions to evaluate the constants c_1 and c_2 we have the following set of simultaneous equations

$$\begin{aligned} c_1 + c_2 e^{\frac{2\bar{\mu}u}{V^2}B} &= -\left(\frac{1}{u}\right)B \\ c_1 + c_2 e^{-\frac{2\bar{\mu}u}{V^2}A} &= \left(\frac{1}{u}\right)A \end{aligned} \quad (27)$$

from which we obtain

$$\begin{aligned} c_1 &= \frac{-B}{2\bar{\mu}u} + \frac{(B+A)}{u}\frac{e^{\frac{2\bar{\mu}uB}{V^2}}}{e^{\frac{2\bar{\mu}uB}{V^2}} - e^{-\frac{2\bar{\mu}uA}{V^2}}} \\ c_2 &= \frac{-B+A}{u} + \frac{1}{e^{\frac{2\bar{\mu}uB}{V^2}} - e^{-\frac{2\bar{\mu}uA}{V^2}}}. \end{aligned} \quad (28)$$

We now consider the specific case when $x_0 = 0$ and $A = B$. After tedious algebraic manipulations we get,

$$\begin{aligned} m_1(0) &= c_1 + c_2 \\ &= \frac{A}{2\bar{\mu}u} tanh\left(\frac{\bar{\mu}uA}{V^2}\right) \end{aligned} \quad (29)$$

For the symmetric case ($\mu_1 = \mu_2$), $u = 0$. Then Eq. (25) becomes

$$\frac{d^2 m_1}{dx_0^2} = -\frac{-2\bar{\mu}}{V^2} \quad (30)$$

whose solution is

$$E(\tau|x_0) = m_1(x_0) = -\frac{\bar{\mu}x_0^2}{V^2} + r_1 x_0 + r_2, \quad -B < x_0 < A \quad (31)$$

Using the boundary conditions we get $r_1 = \frac{(A-B)\bar{\mu}}{V^2}$ and $r_2 = \frac{\bar{\mu}B^2}{V^2} + \frac{B(A-B)\bar{\mu}}{V^2}$. Therefore, for $x_0 = 0$ and $A = B$, $E(\tau) = \frac{\bar{\mu}A^2}{V^2}$.

The expected time for the sequential test to terminate, $E(\tau)$, gives a measure of the amount of transmitted power required for noisy communication system to achieve the given false alarm and miss probability of errors [19]. The faster the test terminates the lesser transmitted power it requires. Therefore, using sequential tests leads to a big savings in the transmitted power. From Eq. (29) for thresholds equidistant from the origin we see that an upper bound on the average stopping time for the test is given by

$$\begin{aligned} E(\tau) &= \frac{A}{2\bar{\mu}u} tanh\left(\frac{\bar{\mu}uA}{V^2}\right) \\ &\leq \frac{A}{2\bar{\mu}|u|} \quad (\text{since } |tanh(.)| \leq 1) \\ &\leq \frac{1}{|\mu_2 - \mu_1|}\left(\frac{A}{V}\right) \end{aligned} \quad (32)$$

IV. Conclusions

Continuous-time sequential decision feedback communication is investigated. The case when the detector test statistic is a Poisson random walk process with reversals whose velocities are governed by a Markov jump process is addressed. Performance of the sequential detector is studied via a restricted stochastic process between two absorbing boundaries. The pdf of the restricted process is derived using the method of Fourier series expansion. The moments of the stopping time of the restricted process is shown to satisfy an ordinary differential equation. From this, the expected stopping time is derived explicitly and an upper bound is also given.

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