S-Modular Games and Power Control in Wireless Networks

Eitan Altman and Zwi Altman

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Goals

- show how centralized or distributed power control algorithms in wireless communications can be viewed as S-modular games
- establish convergence results of power control algorithms
- obtain new properties of S-modular games from results that have been obtained for power control problems

S-modular games

- Refers to Super-modular and Sub-modular
- Super-modular game:
  - if the other players increase their strategy, it pays for you to increase yours
  - strategy space S_i is a sublattice of $\mathbf{R}_m$
    - i.e. S_i has a component-wise least element (a.k.a. meet, a.k.a. x^y) and a component-wise greatest element (a.k.a. join, a.k.a. x∨y)
  - u_i is supermodular in s_i

    $$u_i(x \wedge y) + u_i(x \vee y) \geq u_i(x) + u_i(y)$$ for any x and y in S
  - for u_i twice differentiable, supermodularity is equivalent to

$$\frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} \geq 0$$

- Sub-modular game:
  - if the other players decrease their strategy, it pays for you to decrease yours
  - replace ≥ with ≤

Important properties of S-Modular Games

- Nash equilibrium exists
- NE can be obtained using a greedy best-response-type algorithm
- best response policies are monotonic in other players’ policies

Constrained Nash Equilibrium

- A joint policy x* is a constrained Nash Equilibrium if for each player i

$$x_i^{*} \in \arg opt_{x_i \in S_i(x^{*}_{-i})} u_i(x_i, x^{*}_{-i})$$

- *opt* = max for supermodular, min for submodular

- ie, the strategy for which all players achieve the maximum utility given the (possibly competing) strategies of the other players

Best Response

- For a supermodular function u

$$BR_i^*(x_{-i}) = \max[\arg\max_{x_i \in S_i(x_{-i})} u(x_i, x_{-i})]$$

- Monotonicity

$$x_{-i} \le x'_{-i} \Rightarrow BR_i^*(x_{-i}) \le BR_i^*(x'_{-i}) \Rightarrow S_i(x_{-i}) \prec S_i(x'_{-i})$$

where $A \prec B$ means for 2 sublattices A and B, the minimum of both sets is in A and the maximum is in B; the **ascending property**

GUA

- General Updating Algorithm
- There are N infinite increasing sequences $\{T^i_k\}$, $i=1..N$, $k=1,2,3,\ldots$
- Player i uses at time T^i_k the best response policy the best response policy to the policies used by all other players just before T^i_k

Note: includes strategic form and extensive form games (playing simultaneously or taking turns)

Theorem

- Assume that for all i=1,…,N, $S_i(\cdot)$ are compact for all values of their argument, and are lower semicontinuous in its argument. Either
    - the game is supermodular, players maximize, and $S_i(\cdot)$ are ascending
    - the game is submodular, players minimize, and $S_i(\cdot)$ are descending
- The following hold:

Theorem, Cont’d

- i An equilibrium exists
- ii If each player *i* initially uses its lowest policy in S_i, or if each player *i* initially uses its largest policy in S_i, then the GUA converges monotonically to an equilibrium.
  - – Monotonicity is the same direction for all players: the sequences of strategies either all increase or all decrease
- iii If we start with a feasible policy **x**, then the sequence of best responses monotonically converges to an equilibrium

Equilibrium Uniqueness

- There exists a unique dominating element in the set of equilibria S* (component-wise)
  - ex. submodular game: if x_1 and x_2 are in S, then x' := x_1 ^ x_2 is feasible (due to the descending property
- The unique minimum is attainable using GUA (for any order of updates) starting at the minimal elements of S_i
  - at each iteration of best responses, the new actions are no greater than the best responses to x_{min}, which is again x_{min}, so their limit is bounded by x_{min}

Uplink Power Control Problem Statement (Yates)

- N users, M base stations, common radio channel
- transmission power of each user must be greater than some level $\mathbf{I}_j(\mathbf{p})$ that depends on the transmission powers of all users **p**
- We wish to obtain **p*** such that

$$p_i^* = \min\{p_i \in S_i s.t. p_i \geq I_i(\mathbf{p}^*)\}$$

where S_i is the set of positive real numbers (unbounded, but can also restrict)

Properties of $I(\mathbf{p})$

- Positivity: $I(\mathbf{p}) > 0$
- Monotonicity: if $\mathbf{p}' \geq \mathbf{p}'$ then $I(\mathbf{p}) \geq I(\mathbf{p}')$
- Scalability: For $a>1$, $aI(\mathbf{p}) \geq I(a\mathbf{p})$

Problem Statement using Game Theory

- “Strategic normal form”
- N players i=1,..,N
- Pure strategy space S_i = $\mathbf{R}_m$
- Utility of each player is a function of transmission power
- Minimization performed for each mobile

The Heart of the Paper

- Assume that there is a feasible solution **p**’ to **p** ≥ I(**p**).
- Then
  - i) there exists a fixed point to the equation **p** = I(**p**);
  - ii) if the sets S_i are discrete, then GUA converges for initial powers of all users corresponding to the lowest or to the largest powers;
  - iii) if the sets S_i are convex (continuous action spaces) then there is a unique fixed point to the equation **p** = I(**p**). Moreover, GUA converges for any initial policy.

Proof

- Feasibility of **p**' implies a**p**' is feasible for all a ≥ 1 (scalability of I)
- Consider $\mathbf{p}_0$, not necessarily feasible
- Consider the following submodular game:
  - actions of player j is power $x_j=p_j$
  - utility of player j is $u(p_j) = p_j$
  - constraint policy set of player j is
    - $S_j(p_{-j}) = \{ p_j : p_j \geq I_j(p), 0 \leq p_j \leq ap'_j \}$
    - a subset of the compact set $S_j = \{ p_j : 0 \leq p_j \leq ap'_j \}$

Proof, cont’d

- Cost of every player j is trivially submodular (depends only on j, so derivative = 0)
- policy set of each user is a compact sublattice
- monotonicity of I implies the decending property
- Sj nonempty for all j

Proof, cont'd

- in case of continuous strategy spaces, need to check that S_j are lower semicontinuous
  - $S_j(.)$ are nonempty convex sets
  - Let $\mathbf{p} = \mathbf{p}^1 \vee \mathbf{p}^2$.
  - Choose the smallest $a \geq 1$ such that $\mathbf{p}/a \leq a\mathbf{p}'_j$
  - Monotonicity and scalability of I imply
    - $aI(p/a) \geq I(\mathbf{p}) \geq I(\mathbf{p}/a)$
  - Monotonicity and scalability of I imply
    - $I(\mathbf{p})/a \leq I(\mathbf{p}/a) \leq I(pi) \leq I(p)$
    - $|I(\mathbf{p}^1) - I(\mathbf{p}^2)| \leq I(\mathbf{p})(1-1/a)$
  - If $\mathbf{p}^2$ tends to $\mathbf{p}^1$ then a tends to 1 which shows the continuity of I

Power Contstraints

- solution might converge to a point that is not feasible for some mobiles
- instead minimize $f(p_j, I_j(p))$, where f is some increasing function on the absolute value of the difference between the arguments