

Game theory for wireless networks

Lecture 5

Outline of the lecture

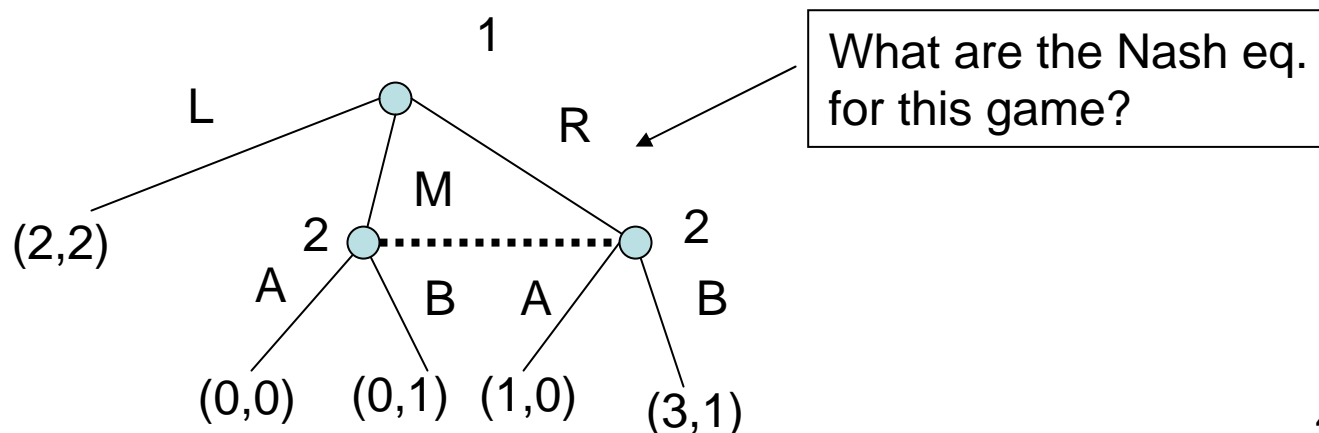
- Dynamic games of incomplete information
 - Perfect Bayesian equilibrium
 - Sequential equilibrium
- Learning in games:
 - Fictitious play

Dynamic games of incomplete information

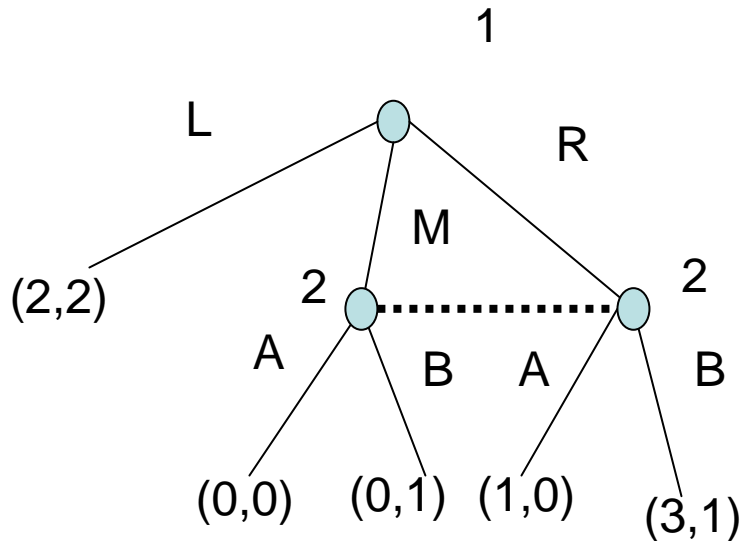
- Recall: static games of incomplete information
 - The game/payoffs depend on the type of players. A player knows its own type but it does not know the types of the other players.
 - Transform a game of incomplete information → game of imperfect information
 - Assign probabilities for the types of the players
 - Perceived as a move by nature
 - Represents the players' *apriori* belief on the types of other players
- What changes for the dynamic game?
 - Players have the chance of updating their beliefs based on the observed actions of the other players.

Subgame perfection for dynamic games of incomplete information?

- The concept of subgame perfection \rightarrow harder to apply for games of incomplete information
 - Start of a period does not form a well-defined subgame
 - Formally: the only proper subgame of a game of incomplete information is the whole game \rightarrow any Nash equilibrium is subgame perfect
 - To illustrate that: consider the following example of a game of imperfect information



Subgame perfection



- Nash equilibria: (L,A) and (R,B)
- Both subgame perfect

However, (L,A) is a sub-optimal action \rightarrow may be ruled for the case of sequential equilibrium

Perfect Bayesian eq. for multi-stage games

- **The Basic Signaling Game**

- Simplest type of game: 2 players

- Player 1: leader (sender)
- Player 2: follower (receiver)
- Player 1 has private inf. about its type $\theta \in \Theta$, action $a_1 \in A_1$
- Player 2: its type is common knowledge, action $a_2 \in A_2$
- Space of mixed actions: \mathcal{A}_1 and \mathcal{A}_2 , with elements α_1 and α_2
- Utility of player i : $u_i(\alpha_1, \alpha_2, \theta)$
- Player 2 – prior belief about player's 1 type: $p \rightarrow$ common knowledge
- Strategy for player 1: probability distribution $\sigma_1(\cdot | \theta)$ over actions $a_1 \in A_1$ for each type θ
- Strategy for player 2: probability distribution $\sigma_2(\cdot | \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$

Basic signaling game: payoffs

- Payoff for player 1, given its type θ :

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_1(a_1, a_2, \theta)$$

- Player 2's (ex ante – beforehand payoff) to strategy $\sigma_2(\cdot | a_1)$, when player 1 plays $\sigma_1(\cdot | \theta)$:

$$\sum_{\theta} p(\theta) \left[\sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta) \right]$$

- Should we make the decision based on the above computed payoff?

Posterior beliefs

- Player 2, observes the action of player 2 \rightarrow must update its belief on θ , and its choice of action \rightarrow posterior distribution over Θ : $\mu(\cdot|a_1)$
- How to compute $\mu(\cdot|a_1)$?
 - Player 1 actions may depend on its type
 - Let $\sigma_1^*(\cdot|\theta)$ to be player 1 strategy
 - Know $p(\cdot)$, $\sigma_1^*(\cdot|\theta)$ and observe a_1 : use Bayes rule
- Extension of subgame-perfect eq. \rightarrow perfect **Bayesian** eq.
 - Player 2 max. its payoff conditional on a_1 :

$$\sum_{\theta} \mu(\theta | a_1) u_2(a_1, \sigma_2(\cdot | a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta | a_1) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta)$$

Perfect Bayesian Equilibrium (PBE)

- **Definition:** A perfect Bayesian eq. of a signaling game is a strategy profile σ^* and posterior beliefs $\mu(\cdot|a_1)$, s.t.:

- (P1): $\forall \theta, \sigma_1^*(\cdot|\theta) \in \arg \max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta)$

- (P2): $\forall a_1, \sigma_2^*(\cdot|a_1) \in \arg \max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1) u_2(a_1, \alpha_2, \theta)$

- (B)
$$\mu(\theta|a_1) = \frac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta')}, \quad \text{if } \sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') > 0$$

$\mu(\cdot|a_1)$, is any prob. distr. on Θ , if $\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') = 0$

Backward induction solution?

- Backward induction was used in games with perfect information
- PBE: strategies are optimal given the beliefs and the beliefs are obtained from equilibrium strategies and observed actions
 - Circularity → PBE cannot be determined by backward induction

An example of a signaling game

- **Two period reputation game**

- 2 firms on the market

Period 1: both firms of the market; only firm 1 action a_1

- Actions for firm 1: prey or accommodate

- If prey: firm 2 gets $P_2 < 0$

- If accommodate: firm 2 gets $D_2 > 0$

- Type of firm 1: “sane” or “crazy”

- “crazy” – always prey

- “sane”

- If accommodates: payoff for 1: $D_1 > 0$

- If preys: payoff for 1: $P_1 < D_1 \rightarrow$ prefers to accommodate

- If 2 exits $\rightarrow M_1 > D_1$ (monopoly)

Two period reputation game: cont.

- Period 2 - player 2 selects action a_2 : stay or exit
 - If exits, it gets a 0 payoff, and player 1 gets $M_1 > D_1$
- Assumptions:
 - Player 1 knows his type
 - Player 2 believes that player 1 is sane with probability p
 - δ = discount factor between the two periods
- Building reputation for the sane player
 - Player 1 may try to convince player 2 that he is crazy, to get $M_1 > D_1$ in the second period of the game.

Taxonomy of PBE

- **Separating equilibrium:** the two types of player 1, choose two different actions in period 1

- Firm 2 has complete information for the second period

$$\mu(\theta = \text{sane} \mid a_1 = \text{accomodate}) = 1$$

$$\mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1$$

- **Pooling equilibrium:** the two types of player 1, choose the same action in period 1

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p$$

- **Hybrid (semi-separating equilibria):** the sane type may randomize between preying and accommodating

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p)$$

$$\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1$$

What type of equilibrium? Existence.

- Separating eq. existence: sufficient and necessary condition

$$D_1(1 + \delta) \geq P_1 + \delta M_1 \Leftrightarrow \delta(M_1 - D_1) \leq (D_1 - P_1)$$

- Pooling eq. \rightarrow to enforce exit for player 2. Condition:

$$pD_2 + (1 - p)P_2 \leq 0$$

- If the above two conditions do not hold \rightarrow hybrid PBE

Note: uniqueness of the eq. in this case, is due to the fact that the “crazy” gtype is assumed to always prey.

Multi-stage games with observed actions and incomplete information

- Each player i has type θ_i , and types are independent

$$p(\theta) = \prod_{i=1}^I p_i(\theta_i)$$

- At each period t ($t=0,1,2,\dots,T$), players choose their actions simultaneously, and the actions are revealed at the end of the period
- Players' action set at a date t is type independent
- Behavior strategy: $\sigma_i(a_i | h^t, \theta)$
- Payoffs $u_i(h^{T+1}, \theta)$
- Subgame perfection \rightarrow BNE not only for the whole game, but also for the “continuation game” starting at period t after all possible histories h^t
 - Continuation games \rightarrow proper subgames?
 - No. They do not stem from a singleton information set

Continuation games → true games

- Need to specify the players' beliefs at the start of each continuation game.
- **Definition:** A perfect Bayesian equilibrium is a (σ, μ) that satisfies (P) and (B(i) – B(iv)).

B(i) Posterior beliefs are independent, and all types of player i have the same belief.

- For all θ , t and h^t :

$$\mu_i(\theta_{-i} | \theta_i, h^t) = \prod_{j \neq i} \mu_j(\theta_j | h^t)$$

- even unexpected events will not change the independence assumption for the type of the opponents

Perfect Bayesian equilibrium: cont

- B(ii) Beliefs are updated according to Bayes' rule:
 - For all i, j, h^t , and a_j^t , If there exist

$\hat{\theta}_j$, s.t. $\mu_i(\theta_j | h^t) > 0$, $\sigma_j(a_j^t | h^t, \theta_j) > 0$, then

$$\mu_i(\theta_j | h^t, a^t) = \frac{\mu_i(\theta_j | h^t) \sigma_j(a_j^t | h^t, \theta_j)}{\sum_{\hat{\theta}_j} \mu_i(\hat{\theta}_j | h^t) \sigma_j(a_j^t | h^t, \hat{\theta}_j)}$$

- B(iii) Don't signal what you don't know
 - For all i, j, h^t , and a^t and \hat{a}^t

$$\mu_i(\theta_j | (h^t, a^t)) = \mu_i(\theta_j | (h^t, \hat{a}^t)), \text{ if } a_j^t = \hat{a}_j^t$$

Perfect Bayesian eq. – cont.

- B(iv) All players have to have the same belief about the type of another player
 - Imposed because of the req. of eq. analysis: players have the same belief about each other's strategies.
 - For all θ_k , and h^t

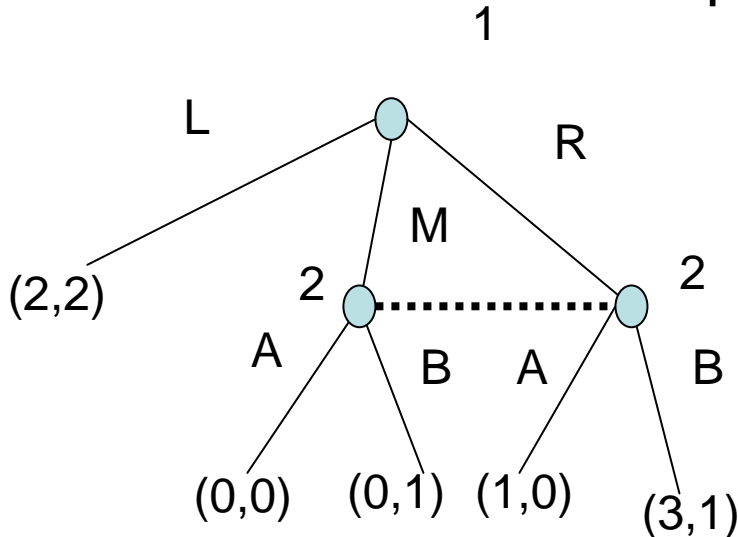
$$\mu_i(\theta_k | h^t) = \mu_k(\theta_k | h^t) = \mu(\theta_k | h^t), \text{ for } i \neq j \neq k$$

- (P) For each player i , type θ_i , alternative strategy σ'_i , and history h^t

$$u_i(\sigma | h^t, \theta_i, \mu(\cdot | h^t)) \geq u_i(\sigma'_i, \sigma_{-i} | h^t, \theta_i, \mu(\cdot | h^t))$$

Sequential equilibrium

- We saw already that the requirement that the players' strategies form a Nash equilibrium is too weak \rightarrow formally the only proper subgame for the games of incomplete, or imperfect information is the whole game.
- Recall the initial example



- Nash equilibria: (L,A) and (R,B)
- Both subgame perfect

Is (LA) equilibrium plausible?

Sequential equilibrium: cont.

- (L,A) is not plausible
 - Whatever player's 2 beliefs on player's 1 move (M or R), he must chose B if he has an opportunity to move.
- Need to generalize the previous condition (P) → given the system of beliefs, no player can gain by deviating at any information set.
- (s) An assessment (σ, μ) is *sequentially rational* if, for any information set h , and alternative strategy $\sigma'_{i(h)}$,

$$u_{i(h)}(\sigma | h, \mu(h)) \geq u_{i(h)}(\sigma'_{i(h)}, \sigma_{-i(h)} | h, \mu(h))$$

Sequential eq. cont.

- Consistency condition on beliefs is also introduced
- (C) An assessment (σ, μ) is consistent if

$$(\sigma, \mu) = \lim_{n \rightarrow +\infty} (\sigma^n, \mu^n)$$

For some sequence $(\sigma^n, \mu^n) \in \Psi^0$ ← The set of all assessments

- **Definition:** A sequential equilibrium is an assessment (σ, μ) that satisfies (S) and (C)

Existence: For any finite extensive-form game, there exist at least one sequential equilibrium.

Learning in games

- Why learning?
 - For introspection, the rules of the game, rationality of the players, payoff functions – all common knowledge
 - Another problem: for multiple equilibria, how players come to expect the same equilibrium?
- Applicability
- Repeated games
- Teach opponent to play a best response to a particular action, by repeating it over and over again

Example of sophisticated learning

- How would you play this game, if you were player 1?

	L	R
U	1,0	3,2
D	2,1	4,0

Sophisticated learning?

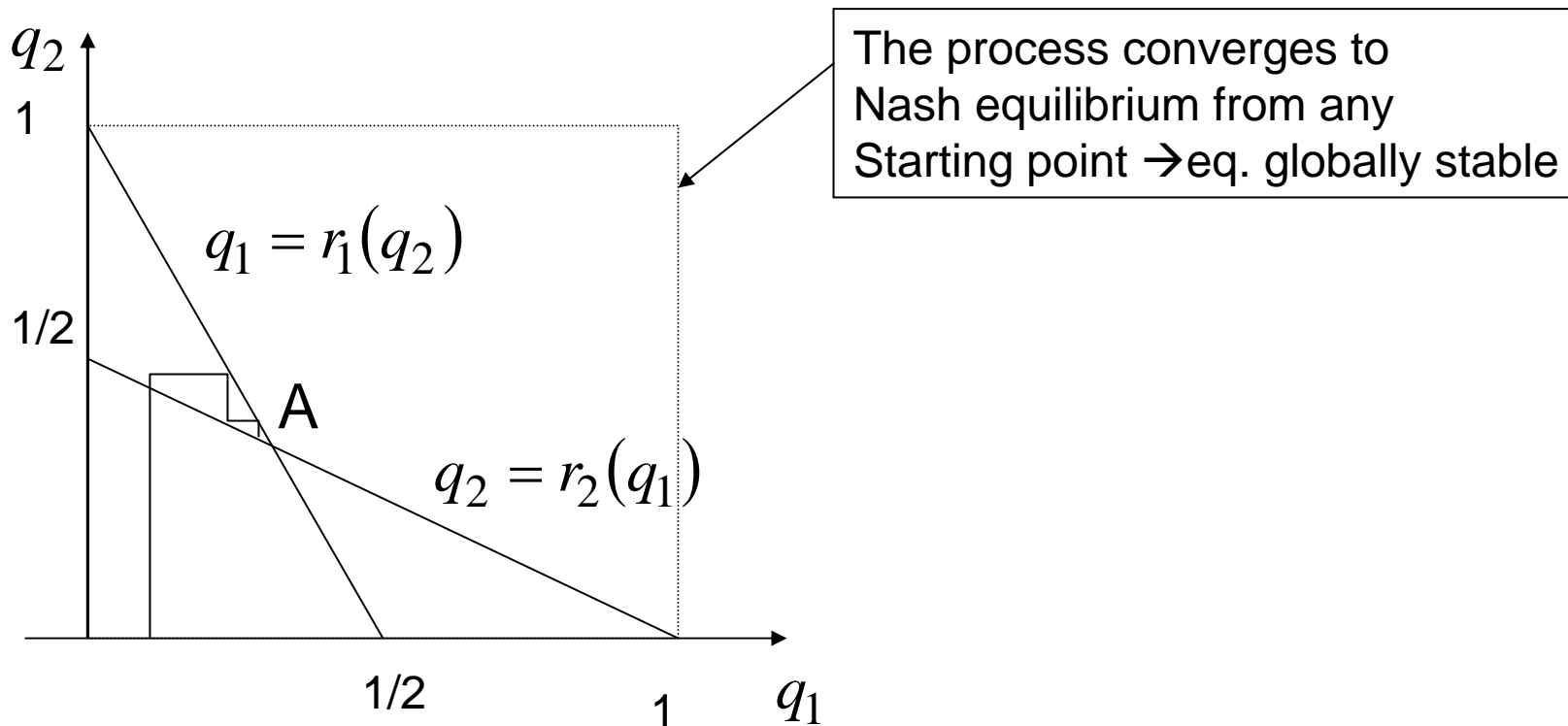
- Most learning theory \rightarrow models for which the incentive is small to alter the future play of the opponents.
 - Examples:
 - large anonymous population: population size large compared to the discount factor
 - Players locked in their choices and discount factor small compared to maximum speed at which the system can possibly adjust

Common models for learning

- **Fictitious play**
 - Players observe only their own matches and play a best response to the historical frequency of play
- **Partial best-response**
 - A fixed portion of users switches each period from its current action to a BR to the aggregate statistics from the previous period
- **Replicator Dynamics**
 - The fraction of the population using a given strategy, grows proportionally to that strategy's current payoff.

One type of learning: Cournot adjustment

- Unique Nash eq. is at the intersection of the reaction curves



Fictitious play

- Repeated game
- Stationary assumption
- Each player: belief of opponents “strategy” by looking at what happened
- Player then plays best response (BR) according to his belief
- Belief: a prediction of the opponent action distribution, i.e. the degree to which player i believes player j will play a certain action.
- Players choose their actions in each period t , s.t. to maximize their expected payoff, with respect to their belief for the current period.

Updating beliefs

- Player i : initial weight function

$$K_0^i : S^{-i} \rightarrow \mathcal{R}^+$$

- Game iteratively repeated \rightarrow K updated:

$$K_t(s^{-i}) = K_{t-1}(s^{-i}) + \begin{cases} 1, & \text{if } s_{t-1}^{-i} = s^{-i} \\ 0, & \text{ow.} \end{cases}$$

- Given the frequency vector $K \rightarrow$ updates beliefs
 - The belief player i has at time t about its opponent to play s^{-i} at time t :

$$\gamma_t^i(s^{-i}) = \frac{K_t^i(s^{-i})}{\sum_{\hat{s} \in S^{-i}} K_t^i(\hat{s}^{-i})} \longleftarrow \text{Simple normalization}$$