

EE800C: Game theory for wireless networks

Lecture 2: January 26 2005

Pareto efficiency example: revisited

| | a_1 | a_2 |
|-------|-------|-------|
| a_1 | 2,3 | -2,7 |
| a_2 | 6,-5 | 0,-1 |

| | a_1 | a_2 |
|-------|-------|-------|
| a_1 | 2,3 | -2,7 |
| a_2 | 6,-5 | 3,5 |



← Pareto efficient

How to get Nash equilibrium?

- Introspection and deduction

Cournot Example

- Duopoly producing a product
- Strategies: quantities produced $q_i \in Q_i = [0, \infty]$
- Price of the goods: $p(q)$; $q = q_1 + q_2$
- Utilities: $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$
- Cournot reaction functions: specify each firm's optimal output for each fixed output level of its opponent

$$r_1 : Q_2 \rightarrow Q_1$$

$$r_2 : Q_1 \rightarrow Q_2$$

$$\text{Max. utility: } p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1))r_2(q_1) - c'_2(r_2(q_1)) = 0$$

Cournot Example cont:

- For $p(q) = \max(0, 1 - q)$
 $c_i(q_i) = cq_i, \quad 0 \leq c \leq 1$

- The reaction functions become

$$r_2(q_1) = (1 - q_1 - c) / 2$$
$$r_1(q_2) = (1 - q_2 - c) / 2$$

- And the Nash equilibrium is:

$$\left. \begin{array}{l} q_1^* = r_2(q_1^*) \\ q_2^* = r_1(q_2^*) \end{array} \right] \Rightarrow q_1^* = q_2^* = (1 - c) / 3$$

How to get to Nash equilibrium?

- Introspection and deduction ← up to now
- **Learning or evolution**
 - Cournot example: players take turns setting their outputs and each player's output is a **best response** to the output his opponent chose in the previous period.

$$q_1^0, q_2^1 = r_2(q_1^0), q_1^2 = r_1(q_2^1) = r_1(r_2(q_1^0)), \dots$$

If it reaches steady state, then

$$\left. \begin{array}{l} q_1^* = r_2(q_1^*) \\ q_2^* = r_1(q_2^*) \end{array} \right] \Rightarrow q_1^* = q_2^* = (1-c)/3 \rightarrow \text{Steady state is a Nash eq.}$$

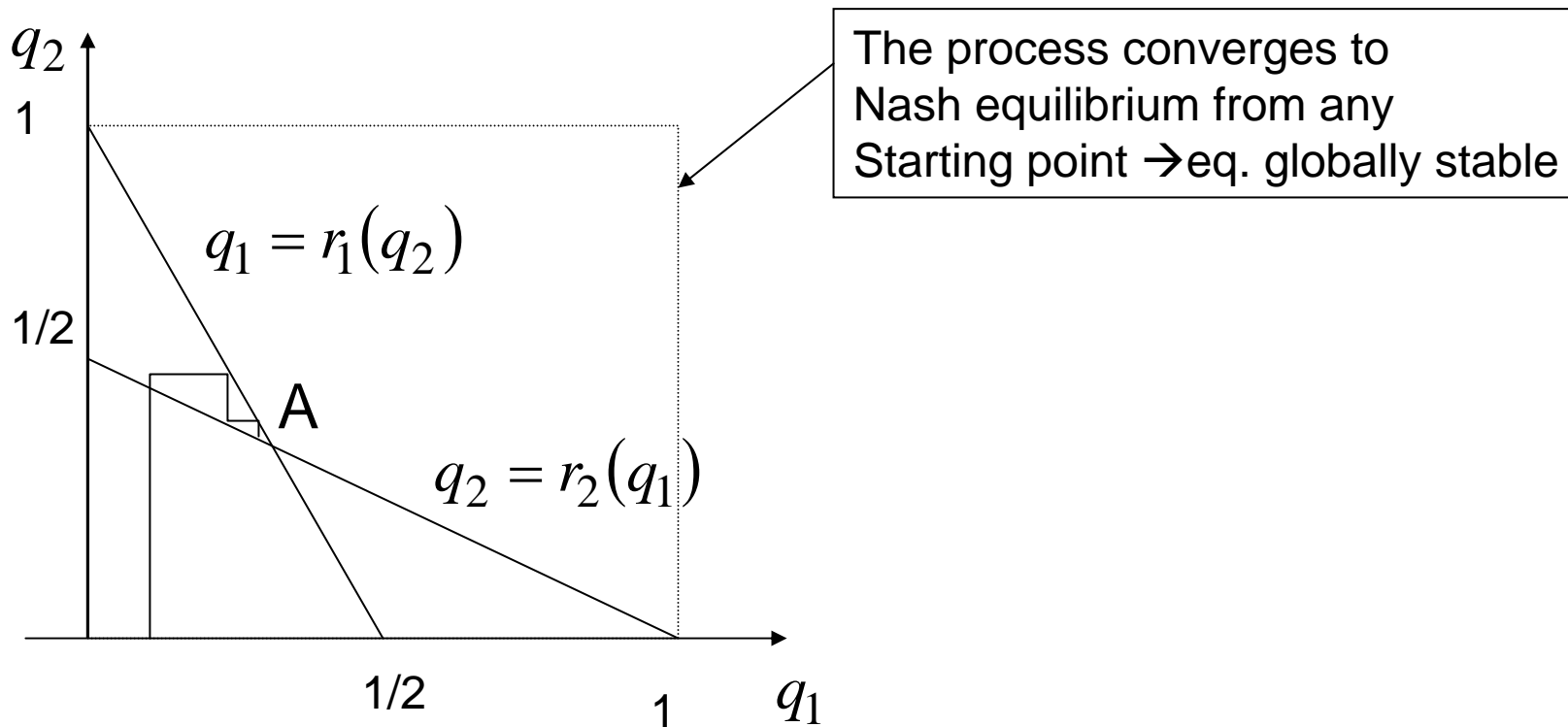
Asymptotically stable equilibrium

- Learning-type adjustment process need not converge to a steady state
- If a process converges to a particular steady state for all initial points sufficiently close to it \rightarrow asymptotically stable
- Cournot example \rightarrow asymptotically stable:

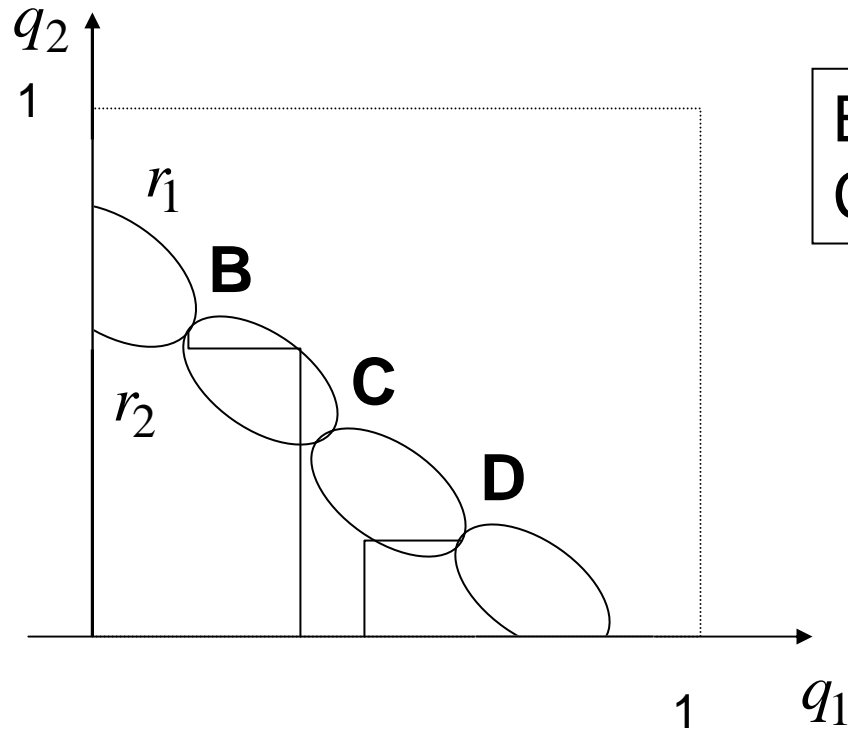
$$\left. \begin{array}{l} p(q) = 1 - q \\ c_i(q_i) = 0 \\ Q_i = [0, 1] \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} r_i(q_j) = (1 - q_j) / 2 \\ \text{Nash eq. } A = \left(\frac{1}{3}, \frac{1}{3} \right) \end{array} \right.$$

Cournot example: evolution

- Unique Nash eq. is at the intersection of the reaction curves



Asymptotic stability: cont.



B, D = stable Nash eq.
C = Nash eq, but not stable

Stability of Nash equilibrium depends on the slope of the reaction functions

Asymptotic stability condition

- Slope of the reaction function:

$$\frac{dr_i}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} / \frac{\partial^2 u_i}{\partial q_i^2}$$

- Sufficient condition for equilibrium (in an open neighborhood of the Nash eq):

$$\left| \frac{dr_1}{dq_2} \right| \left| \frac{dr_2}{dq_1} \right| < 1 \Leftrightarrow \frac{\partial^2 u_1}{\partial q_1 \partial q_2} \frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2} \frac{\partial^2 u_2}{\partial q_2^2}$$

- Note: Same stability condition if the firms react simultaneously instead of alternatively to their opponent's current actions

Example: Shapley cycle

- Best response adjustment process does not necessarily converge

| | L | M | R |
|---|-----|-----|-----|
| U | 0,0 | 4,5 | 5,4 |
| M | 5,4 | 0,0 | 4,5 |
| D | 4,5 | 5,4 | 0,0 |

The table illustrates a Shapley cycle in a 3x3 game. The players are U, M, and D, and the strategies are L, M, and R. The payoffs are as follows:

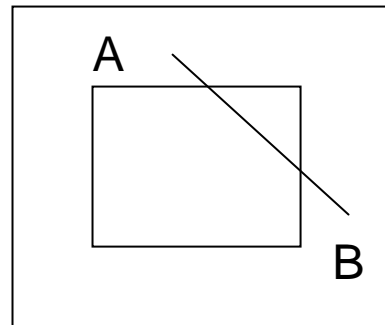
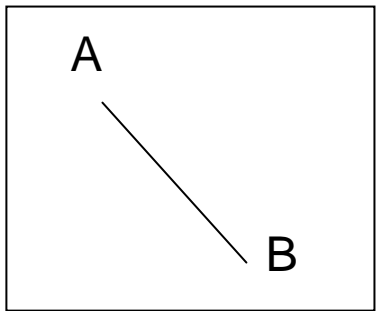
- U chooses M, M chooses L, L chooses D, D chooses M, M chooses R, R chooses U, and U chooses M again.

Adjustment process

- A form of repeated game play
- Players ignore the effect of their current actions on the opponent's future actions
- **Myopic play** = a repeated game in which there is no communication between players, no memory of past events, or prediction of future payoffs. The adaptation is based on the current state of the game.
- Two convergence dynamics possible in a myopic game
 - Best response dynamic
 - Better response dynamic
 - Both require additional conditions to ensure convergence.

Existence of Nash equilibria

- Theorem: Every finite strategic-form game has a mixed strategy equilibrium.
 - Before giving a proof, we need some functional analysis basics
 - **Convex set:** A set S in n -dimensional space is called a convex set, if the line segment joining any pair of points of S lies entirely in S .



Some more functional analysis definitions

- **Compact set:** A bounded set S is compact if there is no point $x \notin S$ such that the limit of a sequence formed entirely from elements in S is x .
 - Compact set \leftrightarrow closed and bounded
 - Examples: any closed finite interval $[a, b]$, closed disc, etc.
 - Not compact: $(a, b]$, $[a, \infty)$

Some more functional analysis definitions

- Continuous function

A function $f: X \rightarrow Y$ is continuous if for all $x_0 \in X$ the following conditions hold:

$$f(x_0) \in Y$$

$$\lim_{x \rightarrow x_0} f(x) \in Y$$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

Quasi-concavity

- A function $f()$ is quasi-concave if

$$f(x) \geq f(y) \Rightarrow f(tx + (1-t)y) \geq f(y)$$

for all x and y and all t between 0 and 1

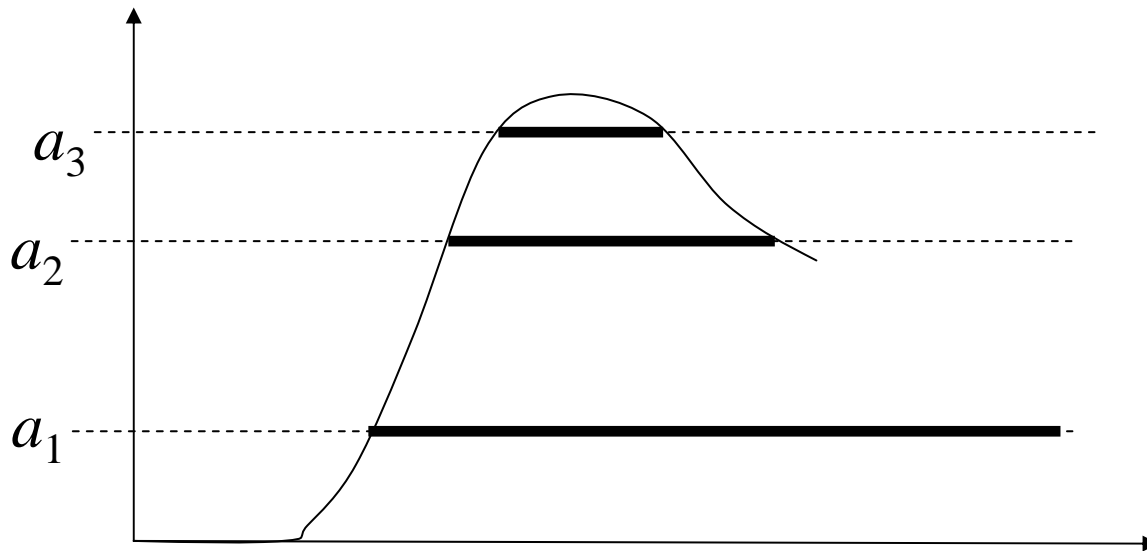
- If $f()$ is a function of one variable and is single-peaked, then $f()$ is quasi-concave

$$\text{Upper Level Set: } P_u = \{x \in X : f(x) \geq a\}$$

- f is quasi-concave if P_u is convex for all a
- All concave functions are quasi-concave
- Any monotonic transformation of a concave function is quasi-concave

Example

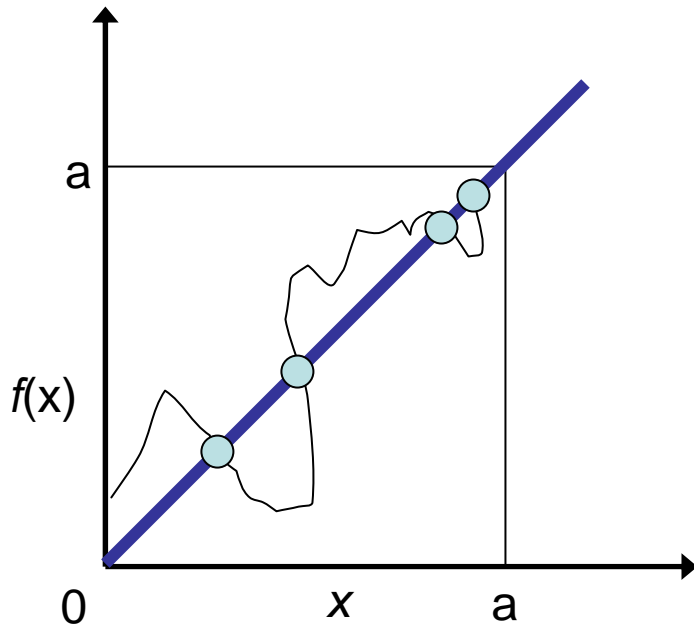
- Some utility function used for power control
- All upper level sets are convex



Fixed point property

- For a given set X , $f : X \rightarrow X$, a fixed point of f is an element x of X such that

$$x = f(x)$$



Existence of fixed point:

A function f can be guaranteed to have a fixed point if it is:

- > Continuous
- > Maps set into itself ($f: X \rightarrow X$)
- > X is compact and convex

Kakutani's fixed point theorem

- Sufficient conditions for $f: X \rightarrow X$ to have a fixed point:
 - (1) X is a compact, convex, nonempty subset of a finite-dimensional Euclidian space
 - (2) $f(x)$ is nonempty for all x
 - (3) $f(x)$ is convex for all x
 - (4) $f(\cdot)$ has a closed graph (equivalent to upper hemi-continuity)

Upper Hemi-continuity: for any x_0 , and for any open set V , that contains $f(x_0)$, There exist a neighborhood U of x_0 , such that $f(x) \subseteq V$, if $x \in U$.

Nash equilibrium existence: revisited

- Theorem: Every finite strategic-form game has a mixed strategy equilibrium.
- Proof is typical; based on the Kakutani fixed point theorem:
- Apply Kakutani to player's reaction functions

$r : \Sigma \rightarrow \Sigma$: cartesian product of r_i . Fixed point for r = Nash eq.

Note: By introducing mixed strategies, the strategy space becomes convex.

Nash equilibrium for infinite games with continuous payoffs

- **Debreu's theorem:** Consider a strategic-form game whose strategy spaces S_i are non-empty, compact, convex subsets of an Euclidian space. If the payoff functions u_i are continuous in s , and **quasiconcave** in s_i , there exists a **pure strategy** Nash equilibrium.
- **Glicksberg theorem:** Consider a strategic-form game whose strategy spaces S_i are nonempty compact subsets of a metric space. If the payoff functions u_i are continuous then there exists a Nash equilibrium in **mixed strategies**.

Homework

Consider a game with 2 players, for which the players have the option of selecting a variable c , with $c \in [0,1]$

The utility associated with each cost selection (their own and their opponents') is defined as

$$u_i(c_i, c_j) = -2c_i^2 + c_j^2 + 3c_i c_j - c_i - 2c_j, \quad i = 1, 2; \quad j \neq i$$

- (a) Does this game have a pure strategy Nash equilibrium?
- (b) Can you implement this game as a learning (evolutionary) game? Determine the reaction functions and analyze the existence conditions for reaching Nash equilibrium for the evolutionary game.
- (c) Repeat (a) and (b) when

$$u_i(c_i, c_j) = -2c_i^2 + c_j^2 + 3c_i c_j + 0.5c_i, \quad i = 1, 2; \quad j \neq i$$