

EE800C: Game theory for wireless networks

Lecture 1 – January 19, 2005

Course information

- Instructor: Cristina Comaniciu, B211,
ccomanic@stevens.edu
- Course materials:
www.ece.stevens.tech.edu/~ccomanic/ee800c.html
- Part 1: Introduction to game theory
 - Textbook: Game Theory, D. Fudenberg and J. Tirole, MIT press 1991
 - Some additional material covered: handouts
- Part 2: Applications for resource allocation in wireless networks
 - Technical papers – seminar like discussions

Course information

- Grading:
 - 20% homework
 - 20% class presentations and discussions
 - 30% midterm
 - 30% project
- Office hours: by appointment

What is game theory?

- A mathematical formalism for understanding, designing and predicting the outcome of games.
- What is a game?
 - Characterized by a number of players (2 or more), assumed to be intelligent and rational, that interact with each other by selecting various actions, based on their assigned preferences.
- Players: **decision makers**
 - A set of actions available for each player
 - A set of preference relationships defined for each player for each possible action tuple.
 - Usually measured by the utility that a particular user gets from selecting that particular action
 - **intelligent** and **rational**

Simple game example: stag or hare

- 2 hunters, have choices hunt stag or hare; successful stag hunting requires cooperation, but it is more rewarding (higher utility)
- Game model

		stag	hare
Actions Player 1	stag	2,2	0,1
	hare	1,0	1,1

What is the outcome of the game?

Equilibrium (Nash equilibrium) = neither player has a unilateral incentive to change its strategy.

Stag or Hare game: Nash equilibrium

Actions player 2

stag hare

Actions Player 1

stag hare

	stag	hare
stag	2,2	0,1
hare	1,0	1,1

One equilibrium is **payoff dominant**

The other is **risk dominant**

Games in strategic (normal) form

- Three elements:
 - The set of players: $\{1, 2, \dots, I\}$ (finite set)
 - The pure strategy space for each player i : S_i
 - Payoff (utility functions) for each profile of strategies:
 $\mathbf{s} = (s_1, \dots, s_I)$

Some notations and definitions:

- $i \rightarrow$ player i 's opponents

Zero sum games – players are indeed pure opponents:

$$\sum_{i=1}^I u_i = 0$$

Note: key feature of zero-sum games – sum of utilities is a constant

More definitions

- Mixed strategies
 - A mixed strategy σ_i is a probability distribution over pure strategies.
 - $\sigma_i(s_i)$ = probability that σ_i assigns to s_i
 - Σ_i = the space of mixed strategies for player i
 - The space of mixed strategy profile: $\Sigma = \times_i \Sigma_i$ with element σ
 - Each player's randomization – statistically independent of those of other players
 - Payoffs for a profile of mixed strategies = expected values of the pure-strategy payoffs:

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=i}^I \sigma_j(s_j) \right) u_i(s)$$

Example mixed strategies

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

$$\sigma_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

$$\sigma_2 = \left(0, \frac{1}{2}, \frac{1}{2} \right)$$

$$u_1(\sigma_1, \sigma_2) = 4 * \frac{1}{3} * 0 + 5 * \frac{1}{2} * \frac{1}{3} + 6 * \frac{1}{3} * \frac{1}{2} + \dots$$

Dominated strategies

- Assumptions:
 - **Common knowledge**: the players know the structure of the strategic game, and know that their opponents know it, and that they know that they know it, etc.
 - **Rationality** - actions taken by a user are in that user's self-interest
- Sometimes, some strategy arises as the best strategy, by a process of elimination: **iterated dominance**
- A **dominant strategy** for a player is a strategy that is better regardless of the actions chosen by the other players

Examples of dominated strategies

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

R gives a higher payoff than M \rightarrow M dominated

If player 1 knows that player 2 does not play M \rightarrow U

If player 2 knows that player 1 chooses U \rightarrow L

Nash Equilibrium is (U,L)

Formal definitions

- **Notations:**

$s_{-i} \in S_{-i}$ ← Strategy selection for all players, but i

$$(s'_i, s_{-i}) = (s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_I)$$

$$(\sigma'_i, \sigma_{-i}) = (\sigma_1, \dots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \dots, \sigma_I)$$

- **Definition:**

- Pure strategy is strictly dominated for player i, if there exists

$\sigma'_i \in \Sigma_i$ such that

$$u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i} \quad (*)$$

- Weakly dominated, if there exists a $\sigma'_i \in \Sigma_i$ such that (*) holds with weak inequality, and the inequality is strict, for at least one s_{-i}

Note: a mixed strategy that assigns positive probability to a dominated strategy, is dominated.

Prisoner's dilemma

- C = cooperate – do not testify
- D = defect - testify

	C	D
C	1,1	-1,2
D	2,-1	0,0

Nash equilibrium

- Many problems (especially in resource allocation) are not solvable by iterated strict dominance
- A broad class of games are characterized by the Nash equilibrium solution.
- Nash equilibrium is a profile of strategies such that each player strategy is an optimal response to the other players' strategies
- Definition: A mixed strategy profile σ^* is a Nash equilibrium, if for all players i ,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*), \quad \forall s_i \in S$$

- Similar def. for the pure strategies

Note: If a player uses a nondegenerate strategy in a Nash eq., he must be Indifferent between all the pure strategies to which he assigns positive probability

Strict Nash equilibrium

- NE is strict if each player has a unique best response to his rivals' strategies
 - No mixed strategies
- s^* is a strict Nash equilibrium iff is a Nash equilibrium, and for all i , and all s_i

$$s_i \neq s_i^* \implies u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$$

Nash equilibrium properties

- Existence
 - Pure strategy Nash equilibrium may not exist
- Uniqueness
 - Nash equilibrium need not to be unique
- Efficiency
 - Pareto Optimality?

Example Games: Matching pennies

- 2 players – simultaneously announce head or tails
- If match – player 1 wins, ow. player 2 wins

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Equilibrium: randomize with prob $1/2$

Example Game: Battle of sexes

Player 1 favors B; player 2 favors F

	B	F
B	1,2	0,0
F	0,0	2,1

An additional mixed strategy equilibrium:

- impose the two choices to be the same for the users
- denote x = probability that player 1 plays F
- denote y = probability that player 2 plays B

Player 1 indif: $0 * y + 2 * (1 - y) = 1 * y + 0 * (1 - y) \Rightarrow y = 2/3$

Player 2 indif: $2 * (1 - x) + 0 * x = 0 * x + 2 * (1 - x) \Rightarrow x = 2/3$

Game: Friend or Foe

- **Game:** At the end of a game show, the winning team may win 1000\$. To win the money, the 2 members of the team must go in separate rooms and vote Friend or Foe. It is assumed that the two team mates are complete strangers. If both choose Friend, then the two players split the money equally. If one chooses Foe, while the other chooses Friend, then the former takes all the money. If both choose Foe, then both players lose, and they get nothing.

Friend or Foe: Nash equilibrium

	Friend	Foe
Friend	500,500	0,1000
Foe	1000,0	0,0

How to get to the Nash Equilibrium?

- Rational Introspection
- Focal Point
- Trial and Error
- Pre-play communication

Which equilibrium is the best?

Pareto efficiency:

An strategy profile is *Pareto optimal* if some players must be hurt in order to improve the payoff of other players

Def: A strategy profile s^* is said to be *Pareto optimal* iff there exists no other strategy profile s' , such that
if for some j

$$u_j(s') > u_j(s^*), \quad u_i(s') \geq u_i(s^*), \quad \forall i \in I \setminus j$$

Observations:

A strategy profile that is a Nash equilibrium may not be Pareto optimal (efficient).

A strategy profile which is Pareto efficient, is not necessarily a Nash equilibrium.

We would like Nash equilibrium to be Pareto efficient.

Example Game

	a_1	a_2
a_1	2,3	-2,7
a_2	6,-5	0,-1

	a_1	a_2
a_1	2,3	-2,7
a_2	6,-5	3,5



← Pareto efficient

Homework

- For the scissors, paper, rock game, the game matrix is depicted below

	scissors	paper	rock
scissors	0,0	1,-1	-1,1
paper	-1,1	0,0	1,-1
rock	1,-1	-1,1	0,0

- Please find the Nash equilibrium solutions for this game.