

# Game theory for wireless networks

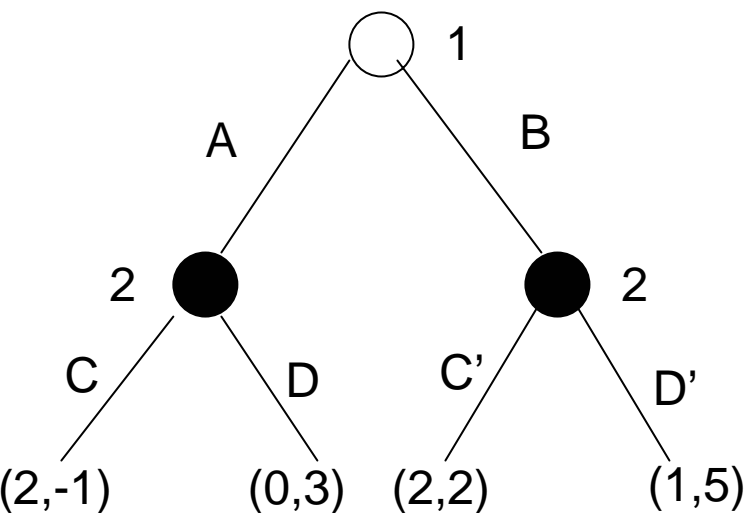
## Lecture 4

# Course outline

- Some observations regarding last lecture's material
- Repeated games: continuation
- Games with incomplete information: Bayesian games

# Some observations: equivalent representation

- Equivalence between extensive form game and normal form games



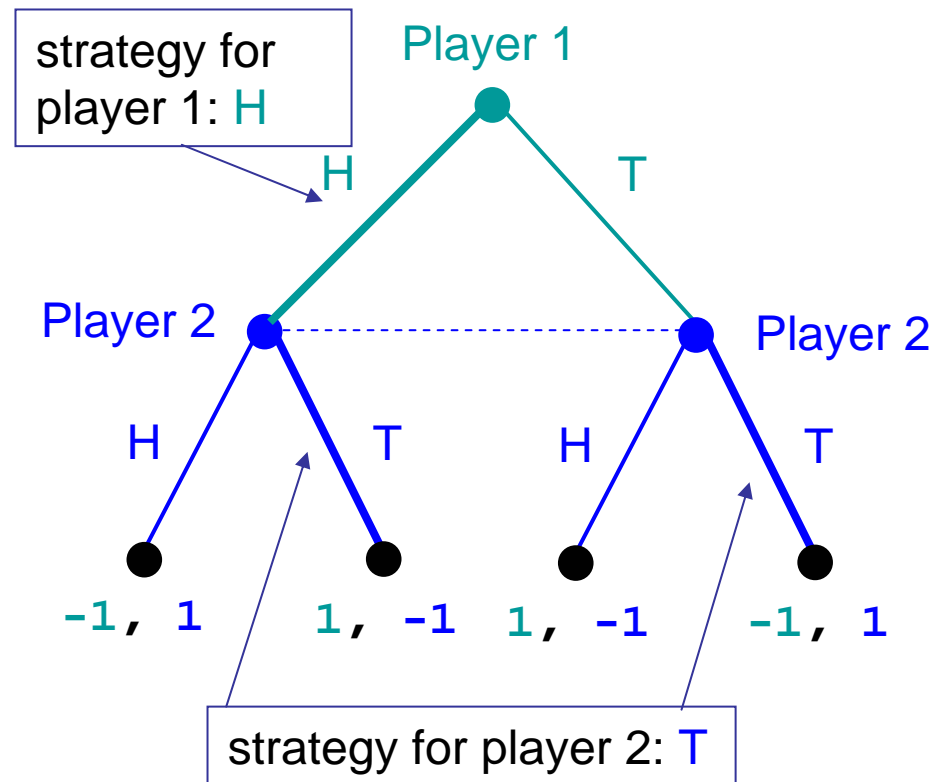
contingent strategies for player 2

	$C, C'$	$C, D'$	$D, C'$	$D, D'$
A	2, -1	2, -1	0, 3	0, 3
B	2, 2	1, 5	2, 2	1, 5

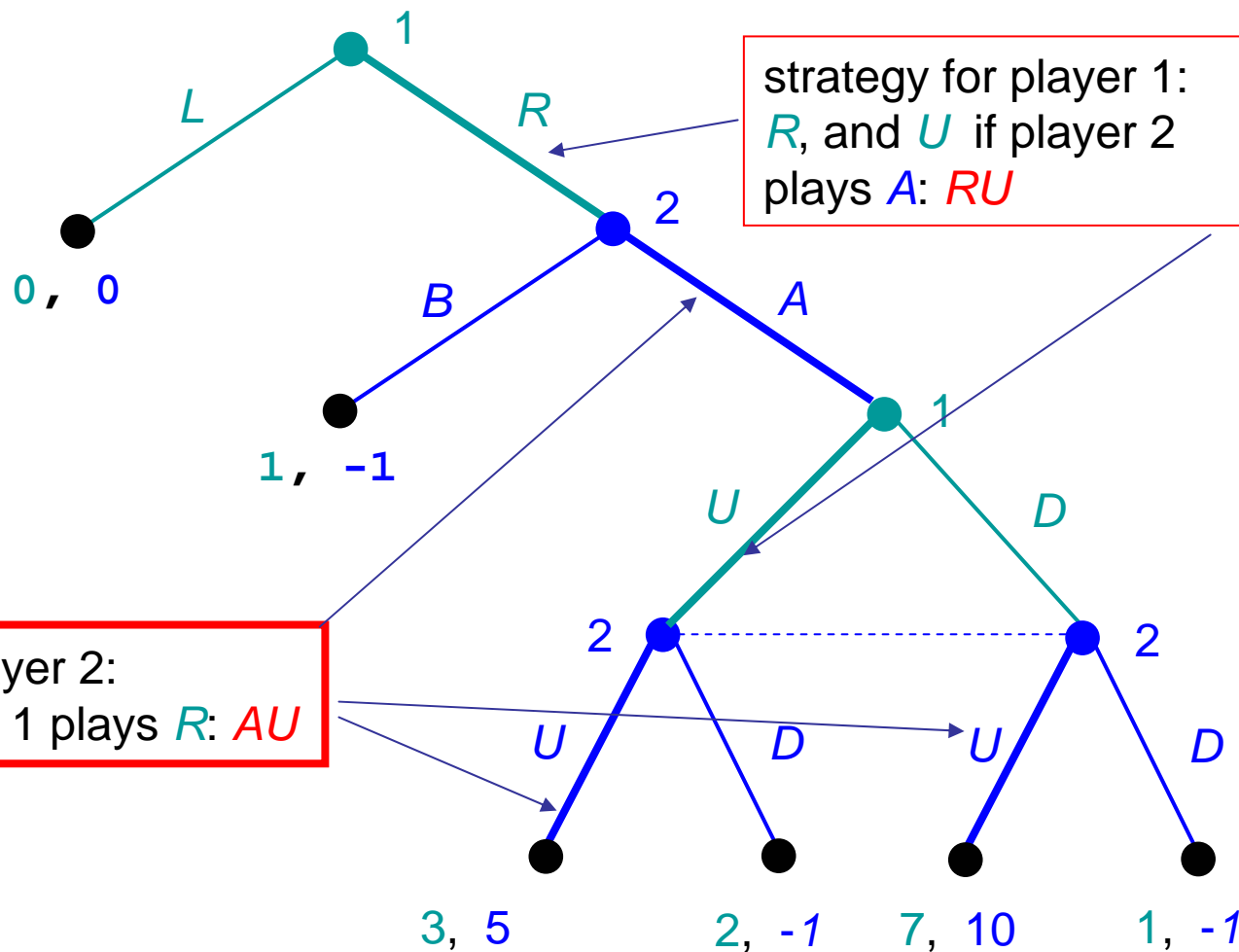
# More clarifications: strategies

- A **strategy** for a player is a **complete plan of actions**.
- It specifies a **feasible action** for the player in **every contingency** that the player might encounter.
- It specifies the action of the player at each of *its information sets*

Matched pennies game example:



# Another example



# More on repeated games

- Last lecture: repeated games may introduce new equilibria and stimulate cooperation
  - Infinitely repeated games (infinite horizon  $T \rightarrow \infty$ )
- Finitely repeated games (finite horizon,  $T$  finite): solved by backward induction
  - Players have incentives to cheat
- Infinite Horizon: description of a game where players think the game extends one more period with high probability
- Finite Horizon: terminal date of the game is known.

# Folk theorem

- Theorem: For every feasible payoff vector  $v$  with  $v_i > \underline{v}_i$  for all players  $i$ , there exist  $\underline{\delta} < 1$ , such that for all  $\delta \in (\underline{\delta}, 1)$ , there is a Nash equilibrium of  $G(\delta)$  with payoffs  $v$ .

$$\underline{v}_i = \min_{\alpha_{-i}} \left[ \max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right] \leftarrow \begin{array}{l} \text{Reservation utility} \\ \text{(minmax value)} \end{array}$$

- If players are sufficiently patient, then any feasible, individually rational payoffs can be enforced as an equilibrium. *In the limit of extreme patience, repeated play allows virtually any payoff to be an equilibrium outcome.*
- *Intuition: when players are patient, any finite one-period gain from deviation is outweighed by even a small loss in utility in every future period.*

# Folk theorem: continuation

- Some definitions:  $\underline{v}_i$  = the lowest payoff, players'  $i$  opponents can hold him to

$m_{-i}^i$  = minimax profile against player  $i$

$$V = \text{convex hull}\{v \mid \exists a \in A, \text{ with } g(a) = v\}$$

- Proof idea: based on “unrelenting strategy”, a player who deviates will be minmaxed in every subsequent period.
  - Assume there exists a pure action profile  $a$ , s.t.  $g(a)=v$
  - Strategy of players: play  $a$  if action  $a$  was played in the previous period, or the action played differed in two or more components; if in the previous period, player  $i$  was the only one to deviate, then play  $m_{-i}^i$  for the rest of the game
  - Question: can player  $i$  gain by deviating from this strategy profile?

Note: convex hull = the smallest convex region enclosing a specified group of points.

# Folk theorem: proof outline

- If player  $i$  deviates at period  $t$ :

$$(1 - \delta^t)v_i + \delta^t(1 - \delta) \max_a g_i(a) + \delta^{t+1} \underline{v}_i$$

- This payoff is less than  $v_i$ , if  $\delta > \underline{\delta}_i$

$$(1 - \underline{\delta}_i) \max_a g_i(a) + \underline{\delta}_i \underline{v}_i = v_i$$

$$\underline{\delta} = \max_i \underline{\delta}_i$$

- Similar arguments for mixed strategy profile (see book)

# Credible equilibrium?

- Strategies used for the proof require all the opponents to play the minmax profile  $\rightarrow$  can be costly  $\rightarrow$  are the opponents threats credible?
- The chosen strategies are not subgame perfect
- Do the conclusions apply to the payoffs of perfect eq.
  - YES. (perfect folk theorem)
- **Nash –threats folk theorem:** Let  $\alpha^*$  be a static equilibrium (an equilibrium for one stage of the game) with payoffs  $e$ . Then for any  $v \in V$  with  $v_i > e_i$ , for all players  $i$ , there is a  $\underline{\delta}$ , such that for all  $\delta > \underline{\delta}$  there is a subgame perfect equilibrium of  $G(\delta)$  with payoffs  $v$ .

# Some other classic strategies for repeated games

Strategy	Description	Advantages and Disadvantages
ALLC	Always cooperate	Susceptible to exploitation
ALLD	Always defect	No cooperation
Tit For Tat (TFT)	Cooperate on the first stage of the game, then do as the other player previously did	Highly robust as a general strategy but when playing against another TFT, cannot recover from an erroneous defection.
Contrite Tit For Tat (CTFT)	Both players start with "good standing." Cooperate if your opponent is in good standing, or if you are not. Otherwise defect.	Maintains a record of an opponent's "standing." Can recover from an opponent's erroneous defection
Generous Tit For Tat (GTFT)	As TFT but cooperate after an opponent's defection with a certain probability	Superior to TFT because it can recover from an erroneous defection. Exploitable by ALLD

# Some other classic strategies for repeated games

Strategy	Description	Advantages and Disadvantages
PAVLOV	Cooperate if and only if both protagonist and opponent played identically in the last round	Adapts by changing strategies when unsuccessful.
Prudent PAVLOV (P-PAVLOV)	Similar to PAVLOV, but only resume cooperation after two rounds of mutual defection	Can recover from an erroneous defection.
REMORSE	Cooperate if in "bad standing" or if both players cooperated in the last round	Maintains a record of an opponents "standing." Can recover from an opponent's erroneous defection
Suspicious Tit For Tat (STFT)	Defect on the first move, otherwise do as the other player last did	If plays against TFT the result is continual defection thereafter

# Some more strategies

<b>Strategy</b>	<b>Description</b>	<b>Advantages and Disadvantages</b>
Tit For Two Tats (TF2T or TFTT)	Cooperate on first move and defect after two consecutive defections by the opponent	Exploitable by a strategy which alternately cooperates and defects
GRIM	Cooperate if both players cooperated previously. Change to ALLD if the other player defects.	Unforgiving. Cannot recover from an erroneous defection

# Games of incomplete information

- Recall:
  - Games of perfect information – sequential games
  - Games of imperfect information – simultaneous move games
  - Games of incomplete information?
    - Some players do not know the payoffs of the others

# Example

- An industry with 2 firms: incumbent (player 1) and potential entrant (player 2)
  - Player 1: Build new plant ?
  - Player 3: Enter?

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

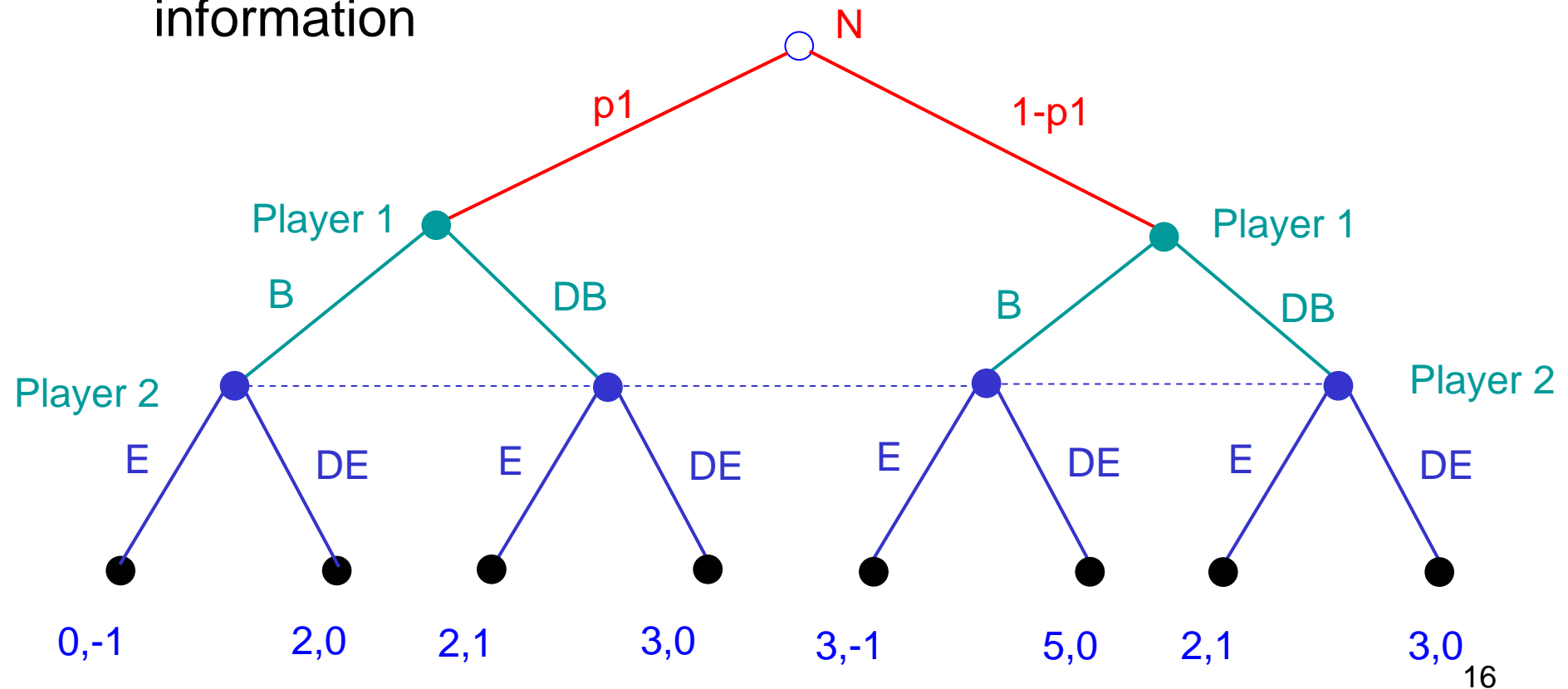
Player 1 knows its cost for building (HIGH or LOW)

Player 2 does not know

→ assign a probability  $p_1$  for HIGH

# Equivalent game

- Introduce prior move by nature: determines player 1 “type”
- Player’s 2 incomplete information  $\rightarrow$  imperfect information



# Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build → dominant strategy for player 1

If cost is LOW; → dominant strategy for player 1: build

How about player 2?

- strategy for 2:

$$\text{enter if : } -1 * (1 - p_1) + 1 * p_1 > 0 \Rightarrow p_1 > 1/2$$

# Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	1.5,-1	3.5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build  $\rightarrow$  dominant strategy for player 1

If cost is LOW; no dominant strategy

$\rightarrow y =$  probability that player 2 enters

$\rightarrow x =$  probability that player 1 builds (given the type of player 1)

Building better than not building:  $1.5y+3.5(1-y) > 2y+3(1-y) \rightarrow y < \frac{1}{2}$

Enter better than not enter:  $(-1) x(1-p_1) + 1[1-x(1-p_1)] > 0$

$\rightarrow x < 1/[2(1-p_1)]$

# Bayesian game characterization

- Type of the players  $\rightarrow$  contains any initial private information that a player might have
- Knowledge about the types is characterized by a pdf

$$p(\theta_1, \theta_2, \dots, \theta_I)$$

$p(\theta_{-i} | \theta_i)$  = player's 1 probability about its opponents types, given its own type

$$p(\theta_i) > 0, \quad \forall \theta_i \in \Theta_i$$

- Given the pure strategy space  $S_i$ , the payoff function of each player  $i$ , will depend on players' types:

$$u(s_1, \dots, s_I, \theta_1, \dots, \theta_I)$$

# Bayesian Equilibrium

- **Definition:** A Bayesian equilibrium in a game of incomplete information with a finite number of types  $\theta_i$  for each player  $i$ , prior distribution  $p$ , and pure strategy spaces  $S_i$  is a Nash equilibrium of the “expanded game”, in which each player’s  $i$  space of pure strategies is the set  $S_i^{\theta_i}$  of maps from  $\theta_i$  to  $S_i$ .

Define a strategy profile:  $s(\cdot)$  and  $s'_i \in S_i^{\theta_i}$

The profile  $s(\cdot)$  is a pure strategy Bayesian equilibrium if, for each player  $i$

$$s_i(\cdot) \in \arg \max_{s'_i(\cdot) \in S_i} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

$$s_i(\cdot) \in \arg \max_{s'_i(\cdot) \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} | \theta_i) u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

# Example: Cournot competition

- Firms select the quantities of production  $s_i = q_i$
- Payoffs defined as

$$u_i = q_i(\theta_i - q_i - q_j)$$

- Common knowledge: firm 1 has type  $\theta_1 = 1$
- Firm 2 – private information about  $\theta_2$
- Firm one beliefs:

$$\theta_2 = 3/4, \quad p = 1/2$$

$$\theta_2 = 5/4, \quad p = 1/2$$

- Belief of Firm 1 is common knowledge
- Firms choose their outputs simultaneously

# Cournot competition: equilibrium

- $q_1$  = firm one's output
- For firm 2: for  $\theta_2 = 3/4$ ,  $q_2^L$   
 $\theta_2 = 5/4$ ,  $q_2^H$

$$\frac{\partial u_2}{\partial q_2} = 0 \Rightarrow q_2(\theta_2) = \frac{\theta_2 - q_1}{2}$$

$$\frac{\partial u_1}{\partial q_1} = 0 \Rightarrow \frac{\partial \left\{ \frac{1}{2} q_1 (1 - q_1 - q_2^H) + \frac{1}{2} q_1 (1 - q_1 - q_2^L) \right\}}{\partial q_1} = 0 \Rightarrow q_1 = \frac{2 - q_2^H - q_2^L}{4}$$

Unique Bayesian equilibrium:  $(q_1 = 1/3; q_2^H = 11/24; q_2^L = 5/24)$

# Another example: Battle of the sexes

Game statement for the imperfect information case:

- Alice and Bob must individually choose to attend either ballet or a football event in the afternoon.
- Common knowledge:
  - ❖ Both would like to spend the afternoon together.
  - ❖ Alice prefers the Ballet.
  - Bob prefers Football.

		Bob	
		B	F
Alice	B	2, 1	0, 0
	F	0, 0	1, 2

# Battle of the sexes with incomplete information

- Bob's preference depends on whether he is happy or not.
- If he is happy  $\rightarrow$  prefers to attend the same event as Alice
- If he is unhappy  $\rightarrow$  prefers to spend the evening alone
- Alice doesn't know if Bob is happy or not. Believes that Bob is happy with probability 0.5 and unhappy with probability 0.5

Bob unhappy

Bob

Alice

B

F

B

F

	B	F
B	2, 0	0, 2
F	0, 1	1, 0

# Battle of the sexes with incomplete information: cont.

- What is the Bayesian Nash equilibrium?

**Bob happy**

		Bob	
		B	F
Alice	B	2, 1	0, 0
	F	0, 0	1, 2

**Bob unhappy**

		Bob	
		B	F
Alice	B	2, 0	0, 2
	F	0, 1	1, 0

# Battle of the sexes with incomplete information: cont

- Best response
  - ❖ If Alice chooses **B** → Bob's best response: **B** if he is happy, and **F** if he is unhappy
  - ❖ After Bob's choice, what is the best response for Alice?
    - ❖ Expected payoff for Alice if her choice is **B**:  
 $2 \times 0.5 + 0 \times 0.5 = 1$
    - ❖ Expected payoff for Alice if her choice is **F**:  
 $0 \times 0.5 + 1 \times 0.5 = 0.5$
    - ❖ Best response for Alice is **B**
  - ❖ (**B**, (**B** if happy and **F** if unhappy)) is a Bayesian Nash equilibrium

# Battle of the sexes with incomplete information: cont

- Best response
  - ❖ If Alice chooses **F** then Bob's best response: **F** if he is happy, and **B** if he is unhappy
  - ❖ After Bob's choice, what is Alice's best response?
    - ❖ Alice expected payoff if she chooses B:  
 $0 \times 0.5 + 2 \times 0.5 = 1$
    - ❖ Alice expected payoff if she chooses F  
 $1 \times 0.5 + 0 \times 0.5 = 0.5$
    - ❖ Since  $1 > 0.5$ , Alice's best response is **B**
  - ❖ (**F**, (**F** if happy and **B** if unhappy)) is not a Bayesian Nash equilibrium.

## Study last year's homework (solutions on the last year's website)

The homework consists of the following three problems:

- 1) Consider the stage game characterized by the following matrix, and assume that the game is going to be played repeatedly:

	A	B	C
D	4,4	1,1	0,0
E	1,1	1,1	0,0
F	0,0	0,0	0,0

- a) Identify the Nash equilibrium
  - b) Identify the minmax strategy profile
- 2) An investor with a wealth of 100, can invest into a business that may double his wealth with probability  $p$ , and may reduce its wealth to  $\frac{1}{2} \cdot 100$ , with probability  $1-p$ .
    - (a) For what values of  $p$  he should invest in the new business, if his payoff function is  $u(x) = \ln(x)$ ?
    - (b) Same question for  $u(x)=x$ .

# Homework problems: cont.

- 3) In a generic game with two players, the payoffs of the players depend on the state of the game. State A appears with probability  $A$ , and state B, appears with probability  $(1-p)$ . The payoff matrices are represented below.

	L	R
U	3,2	1,0
D	1,-2	2,0
	<b>A</b>	

	L'	R'
U'	1,2	0,0
D'	0,-1	1,0
	<b>B</b>	

- (a) Show that  $(L,R')$  can never be a best response for player 2.
- (b) Can the pure strategy  $(L,L')$  be used in a Bayes-Nash equilibrium? If so, what does player 1 choose?
- (c) Same question for  $(R,R')$ .
- (d) Same question for  $(R,L')$ .

Hint: assume that player 1 plays U with probability  $q$ , then analyze the various cases.