EE800C: Game theory for wireless networks

Lecture 2: January 24 2006

Pareto efficiency: revisited

Pareto efficiency:

A strategy profile is *Pareto optimal* if some players must be hurt in order to improve the payoff of other players.

Def: A strategy profile *s** is said to be *Pareto optimal* iff there exists no other strategy profile *s'*, such that if for some *j*

$$u_j(s') > u_j(s^*), \quad u_i(s') \geq u_i(s^*), \ \forall i \in I \setminus j$$

How to get Nash equilibrium?

- Introspection and deduction

Cournot Example

- Duopoly producing a product
- Strategies: quantities produced $q_i \in Q_i = [0, \infty]$
- Price of the goods: $p(q)$; $q = q_1 + q_2$
- Utilities: $u_i(q_1, q_2) = q_i p(q) - c_i(q_i)$

cost of production

- Cournot reaction functions: specify each firm's optimal output for each fixed output level of its opponent

$$r_1 : Q_2 \rightarrow Q_1$$
$$r_2 : Q_1 \rightarrow Q_2$$

Max. utility: $p(q_1 + r_2(q_1)) + p'(q_1 + r_2(q_1)) r_2(q_1) - c'_2(r_2(q_1)) = 0$

Cournot Example cont:

- For $p(q) = \max(0, 1 - q)$

  $c_i(q_i) = cq_i, \qquad 0 \le c \le 1$

- The reaction functions become

$$r_2(q_1) = (1 - q_1 - c)/2$$
$$r_1(q_2) = (1 - q_2 - c)/2$$

- And the Nash equilibrium is:

$$\left.\begin{array}{l} q_1^* = r_2\left(q_1^*\right) \\ q_2^* = r_1\left(q_2^*\right) \end{array}\right] \Rightarrow q_1^* = q_2^* = (1 - c)/3$$

How to get to Nash equilibrium?

- Introspection and deduction ← up to now
- Learning or evolution
  - Cournot example: players take turns setting their outputs and each player's output is a best response to the output his opponent chose in the previous period.

$$q_1^0, q_2^1 = r_2\left(q_1^0\right), q_1^2 = r_1\left(q_2^1\right) = r_1\left(r_2\left(q_1^0\right)\right), \ldots$$

If it reaches steady state, then

$$\left.\begin{aligned} q_1^* &= r_2\left(q_1^*\right) \\ q_2^* &= r_1\left(q_2^*\right) \end{aligned}\right] \Rightarrow q_1^* = q_2^* = (1-c)/3$$ → Steady state is a Nash eq.

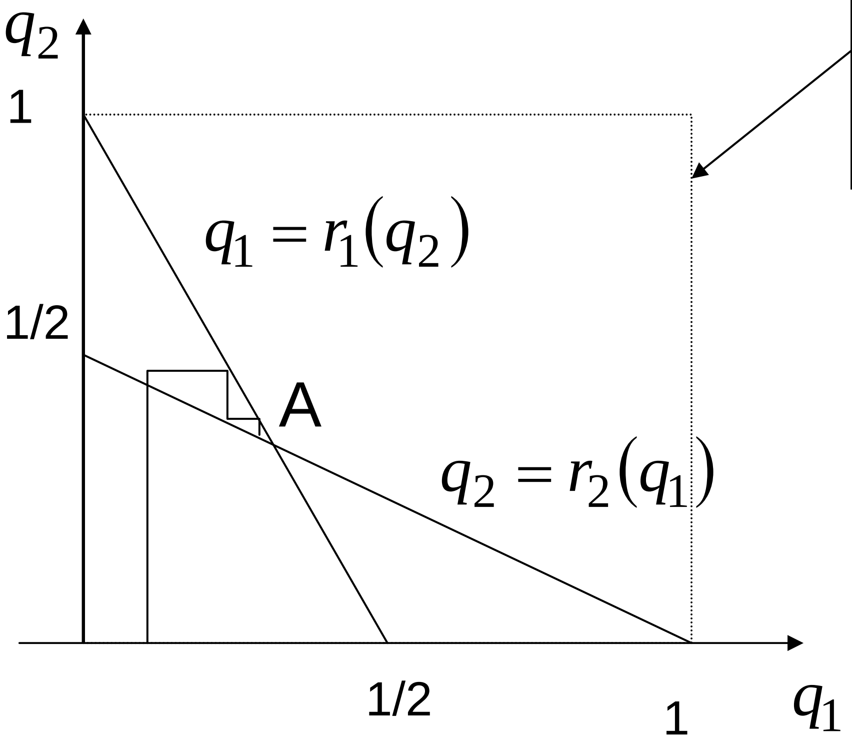
Asymptotically stable equilibrium

- Learning-type adjustment process need not converge to a steady state
- If a process converges to a particular steady state for all initial points sufficiently close to it → asymptotically stable
- Cournot example → asymptotically stable:

$$\left.\begin{array}{l} p(q) = 1 - q \\ c_i(q_i) = 0 \\ Q_i = [0,1] \end{array}\right] \Rightarrow \left\{\begin{array}{l} r_i(q_j) = (1 - q_j)/2 \\ \text{Nash eq. } A = \left(\frac{1}{3}, \frac{1}{3}\right) \end{array}\right.$$

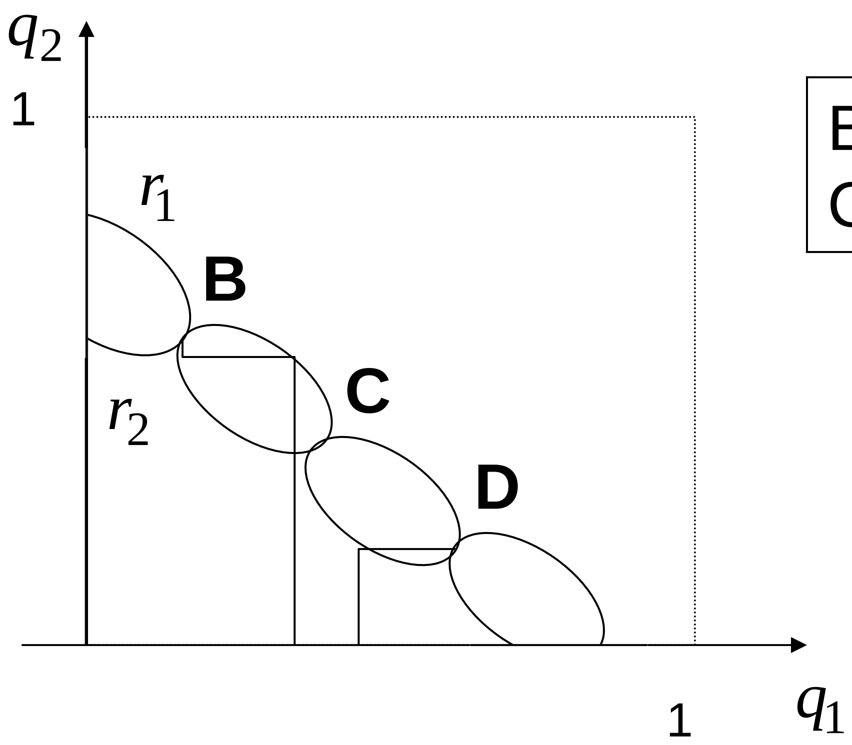
Cournot example: evolution

- Unique Nash eq. is at the intersection of the reaction curves



The process converges to Nash equilibrium from any starting point →eq. globally stable

Asymptotic stability: cont.



B, D = stable Nash eq.
C = Nash eq, but not stable

Stability of Nash equilibrium depends on the slope of the reaction functions

Asymptotic stability condition

- Slope of the reaction function:

$$\frac{dr_i}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} / \frac{\partial^2 u_i}{\partial q_i^2}$$

- Sufficient condition for equilibrium (in an open neighborhood of the Nash eq):

$$\left|\frac{dr_1}{dq_2}\right|\left|\frac{dr_2}{dq_1}\right| < 1 \Leftrightarrow \frac{\partial^2 u_1}{\partial q_1 \partial q_2}\frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2}\frac{\partial^2 u_2}{\partial q_2^2}$$

- Note: Same stability condition if the firms react simultaneously instead of alternatively to their opponent’s current actions

Example: Shapley cycle

- Best response adjustment process does not necessarily converge

	L	M	R
U	0,0	4,5	5,4
M	5,4	0,0	4,5
D	4,5	5,4	0,0

Adjustment process

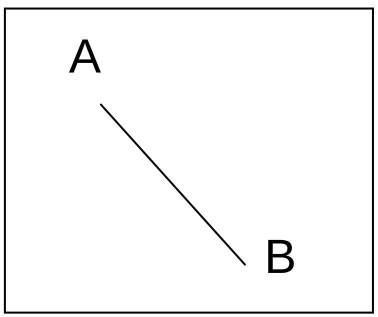
- A form of repeated game play
- Players ignore the effect of their current actions on the opponent's future actions
- Myopic play = a repeated game in which there is no communication between players, no memory of past events, or prediction of future payoffs. The adaptation is based on the current state of the game.
- Two convergence dynamics possible in a myopic game
  - Best response dynamic
  - Better response dynamic
    - Both require additional conditions to ensure convergence.

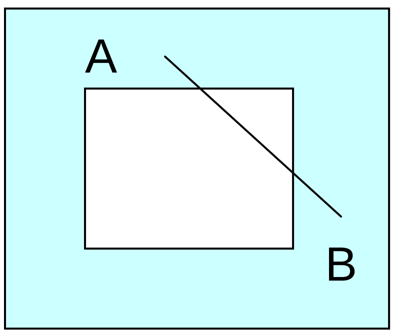
Existence of Nash equilibria

- Theorem: Every finite strategic-form game has a mixed strategy equilibrium.

  – Before giving a proof, we need some functional analysis basics

  – Convex set: A set *S* in n-dimensional space is called a convex set, if the line segment joining any pair of points of S lies entirely in S.





Some more functional analysis definitions

- Compact set: A bounded set S is compact if there is no point $x \notin S$ such that the limit of a sequence formed entirely from elements in S is x.
  - Compact set ←→ closed and bounded
  - Examples: any closed finite interval [a, b], closed disc, etc.
  - Not compact: (a, b], [a, ∞)

Some more functional analysis definitions

- Continuous function

A function f: $X \to Y$ is continuous if for all $x_0 \in X$ the following conditions hold:

$$f(x_0) \in Y$$

$$\lim_{x \to x_0} f(x) \in Y$$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Convex functions

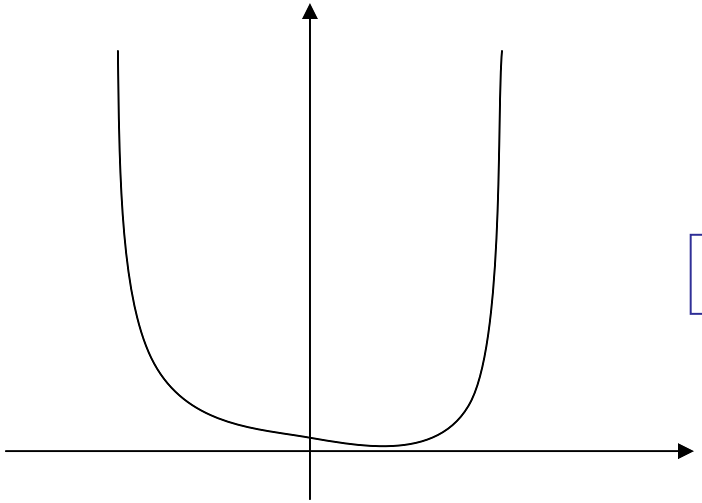
- A convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the average of its values at the ends of the interval.
- f(x) ic convex on [a,b] if for any two points x1, x2$\in$ [a,b], and any 0<λ<1,

$$f\left[\lambda x_1 + (1-\lambda)x_2\right] \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

More on convex functions

- If f(x) has a second derivative on [a, b], then a necessary and sufficient condition for it to be convex is that

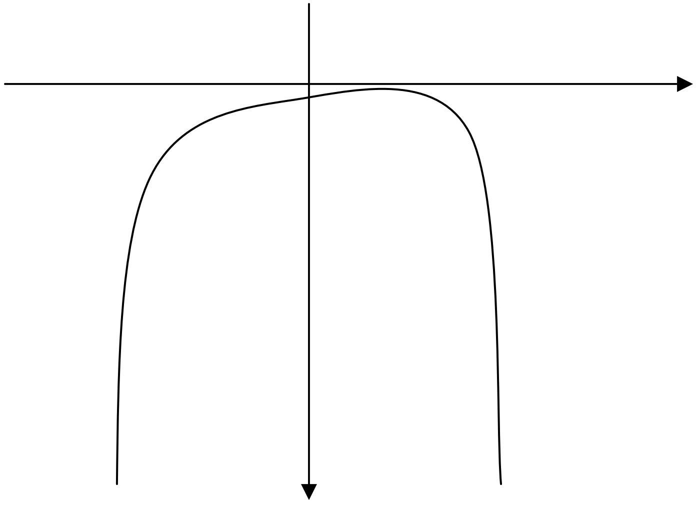
$$f''(x) \geq 0, \forall x \in [a, b]$$



Convex function

Concave function

- $f(x)$ is concave if $-f(x)$ is convex.



Concave function

Quasi-concavity

- A function f() is quasi-concave if

$$f(x) \geq f(y) \Rightarrow f\left(tx + (1-t)y\right) \geq f(y)$$

for all x and y and all t between 0 and 1

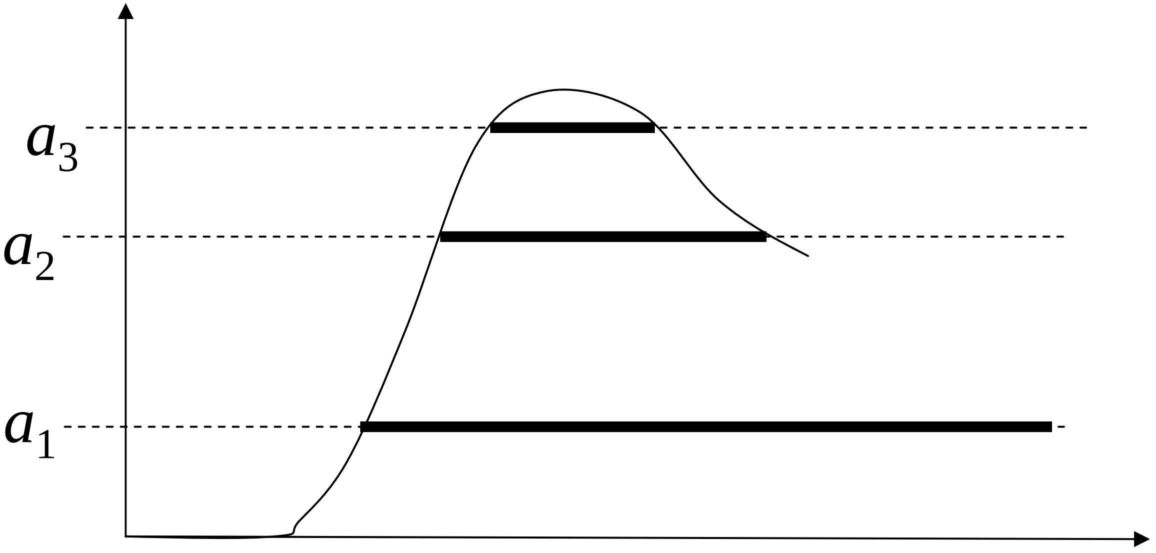
- If f() is a function of one variable and is single-peaked, then f() is quasi-concave

Upper Level Set: $P_u = \left\{ x \in X : f\left(x\right) \geq a \right\}$

- f is quasi-concave if P_u is convex for all a
- All concave functions are quasi-concave
- Any monotonic transformation of a concave function is quasi-concave

Example

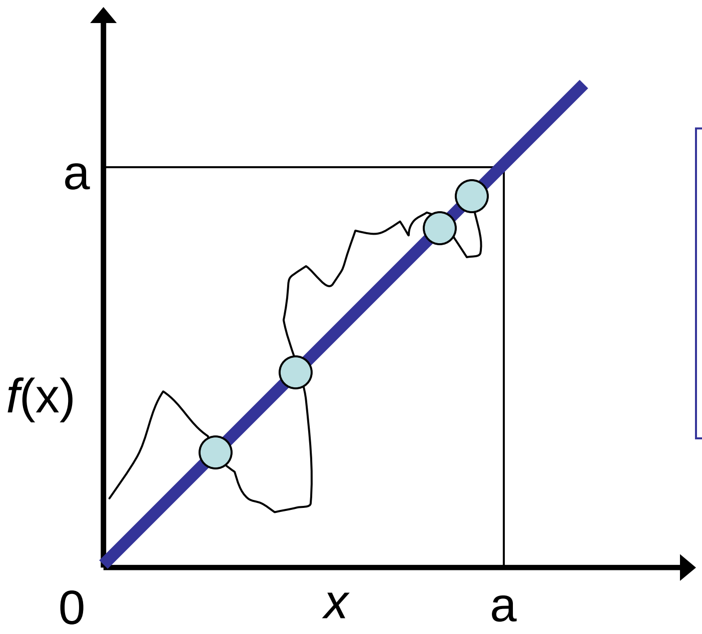
- Some utility function used for power control
- All upper level sets are convex



Fixed point property

- For a given set X, $f : X \rightarrow X$, a fixed point of f is an element x of X such that

$$x = f(x)$$



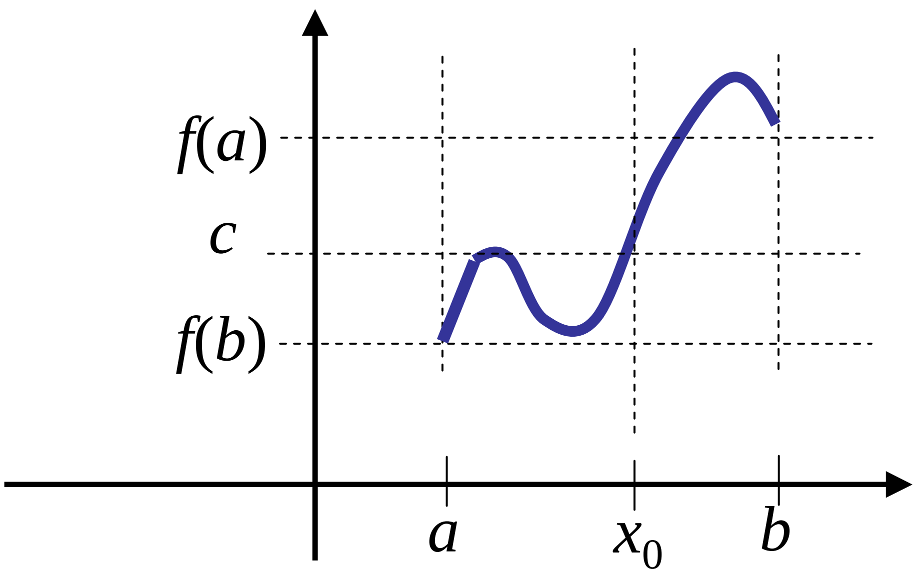
Existence of fixed point:

A function f can be guaranteed to have a fixed point if it is:

> Continuous
> Maps set into itself (f: $X \rightarrow X$)
> X is compact and convex

Justified by the Intermediate Value Theorem

If f is **continuous** on a closed interval $[a,b]$, and c is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x_0 in the closed interval $[a,b]$ such that $f(x_0) = c$.



Kakutani's fixed point theorem

- Sufficient conditions for f: X → X to have a fixed point:
  - (1) X is a compact, convex, nonempty subset of a finite-dimensional Euclidian space
  - (2) f(x) is nonempty for all x
  - (3) f(x) is convex for all x
  - (4) f(.) has a closed graph (equivalent to upper hemi-continuity)

Upper Hemi-continuity: for any *x*0, and for any open set V, that contains f(x0),
There exist a neighborhood U of x0, such that f(x) $\subseteq$ V, if x$\in$ U.

Nash equilibrium existence: revisited

- Theorem: Every finite strategic-form game has a mixed strategy equilibrium.

- Proof is typical; based on the Kakutani fixed point theorem:
- Apply Kakutani to player's reaction functions

$r : \Sigma \rightarrow \Sigma$: cartesian product of ri. Fixed point for r = Nash eq.

Note: By introducing mixed strategies, the strategy space becomes convex.

Nash equilibrium for infinite games with continuous payoffs

- Debreu's theorem: Consider a strategic-form game whose strategy spaces S_i are non-empty, compact, convex subsets of an Euclidian space. If the payoff functions u_i are continuous in s, and quasiconcave in s_i, there exists a pure strategy Nash equilibrium.

- Glicksberg theorem: Consider a strategic-form game whose strategy spaces S_i are nonempty compact subsets of a metric space. If the payoff functions u_i are continuous then there exists a Nash equilibrium in mixed strategies.

Notes

- **Euclidian n-space**, also called cartesian space, or n-space: $\Re^n$
  - Space of all *n*-tuples of real numbers,
  - vector space

$$\left(x_1, x_2, \ldots, x_n\right)$$

- **Metric space**: a set *S* with a global distance function (metric *g*) that for every two points *x, y* $\in$ *S*, gives the distance between them as a nonnegative real number *g(x,y).* The metric must satisfy:
- 1. *g(x,y)=0*, iff *x=y*
- *2.g(x,y)=g(y,x)*
- 3. Triangle inequality: *g(x,y)+ g(y,z)* $\geq$ *g(x,z)*

Homework

- Study homework 2 and its solution from last year's class:

http://www.ece.stevens-tech.edu/~ccomanic/ee800c.html

- Solve new homework 2 assigned.