EE672: Game theory for wireless networks

Lecture 1 – January 18, 2006

Course information

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- Course materials:

www.ece.stevens.tech.edu/~ccomanic/ee672.html

- Part 1: Introduction to game theory
  - Textbook: Game Theory, D. Fudenberg and J. Tirole, MIT press 1991
  - Some additional material covered: handouts
- Part 2: Applications for resource allocation in wireless networks
  - Technical papers – seminar like discussions

Course information

- Grading:
  - 20% homework
  - 20% class presentations and discussions
  - 30% midterm
  - 30% project
- Office hours: by appointment

What is game theory?

- A mathematical formalism for understanding, designing and predicting the outcome of games.
- What is a game?
  - Characterized by a number of players (2 or more), assumed to be intelligent and rational, that interact with each other by selecting various actions, based on their assigned preferences.
- Players: decision makers
  - A set of actions available for each player
  - A set of preference relationships defined for each player for each possible action tuple.
    - Usually measured by the utility that a particular user gets from selecting that particular action
  - intelligent and rational

Simple game example: stag or hare

- 2 hunters, have choices hunt stag or hare; successful stag hunting requires cooperation, but it is more rewarding (higher utility)
- Game model

Actions player 2

Actions Player 1

	stag	hare
stag	2,2	0,1
hare	1,0	1,1

What is the outcome of the game?
Equilibrium (Nash equilibrium) = neither player has a unilateral incentive to change its strategy.

Stag or Hare game: Nash equilibrium

Actions player 2

Actions Player 1

	stag	hare
stag	2,2	0,1
hare	1,0	1,1

One equilibrium is payoff dominant

The other is risk dominant

Which equilibrium point will be played?

Without further information on the game – cannot tell!

Games in strategic (normal) form

- Three elements:
  - The set of players: {1, 2, …, I } (finite set)
  - The pure strategy space for each player i: S_i
  - Payoff (utility functions) for each profile of strategies: $\mathbf{s} = (s_1, \ldots, s_I)$

Some notations and definitions:

-i → player i's opponents

Zero sum games – players are indeed pure opponents:

$$\sum_{i=1}^{I} u_i = 0$$

← constant set to zero → normalization

Note: key feature of zero-sum games – sum of utilities is a constant

More definitions

- Mixed strategies
  - A mixed strategy σ_i is a probability distribution over pure strategies.
  - $\sigma_i(s_i)$ = probability that σ_i assigns to s_i
  - Σ_i = the space of mixed strategies for player i
  - The space of mixed strategy profile: $\Sigma = \times_i \Sigma_i$ with element σ
    - Each player's randomization – statistically independent of those of other players
    - Payoffs for a profile of mixed strategies = expected values of the pure-strategy payoffs:

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=i}^{I} \sigma_j(s_j) \right) u_i(s)$$

Player i's payoff to profile σ

Example mixed strategies

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

$$\sigma_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \quad u_1(\sigma_1, \sigma_2) = 4*1/3*0 + 5*1/2*1/3 + 6*1/3*1/2 + \ldots$$

$$\sigma_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right) \quad u_1(\sigma_1, \sigma_2) = \frac{11}{2} \quad u_1(\sigma_1, \sigma_2) = \frac{27}{6}$$

Dominated strategies

- Assumptions:
  - Common knowledge: the players know the structure of the strategic game, and know that their opponents know it, and that they know that they know it, etc.
  - Rationality - actions taken by an user are in that user's self-interest
- Sometimes, some strategy arises as the best strategy, by a process of elimination: iterated dominance
- A dominant strategy for a player is a strategy that is better regardless of the actions chosen by the other players

Dominant strategy?

- Does the stag/hare game have a dominant strategy?

Actions player 2

Actions Player 1

	stag	hare
stag	2,2	0,1
hare	1,0	1,1

Examples of dominated strategies

Actions Player 2

Actions Player 1

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

M strictly dominated

R gives a higher payoff than M → M dominated
If player 1 knows that player 2 does not play M → U
If player 2 knows that player 1 chooses U → L
Nash Equilibrium is (U,L)

Formal definitions

- **Notations:**

$s_{-i} \in S_{-i}$ ← Strategy selection for all players, but i

$$(s'_i, s_{-i}) = (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_I)$$

$$(\sigma'_i, \sigma_{-i}) = (\sigma_1, \ldots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \ldots, \sigma_I)$$

- **Definition:**
  - Pure strategy s_i is strictly dominated for player i, if there exists $\sigma'_i \in \sum_i$ such that

$$u_i(\sigma'_i, s_{-i}) > u_i(s_i, s_{-i}), \ \forall \ s_{-i} \in S_{-i} \quad (*)$$

  - Weakly dominated, if there exists a $\sigma'_i \in \sum_i$ such that (*) holds with weak inequality, and the inequality is strict, for at least one s_{-i}

Note: a mixed strategy that assigns positive probability to a dominated strategy,
Is dominated.

Prisoner's dilemma

- C = cooperate – do not testify
- D = defect - testify

	C	D
C	1,1	-1,2
D	2,-1	0,0

If the game is played repeatedly → outcome may change
- fear of future consequences if they defect

Nash equilibrium

- Many problems (especially in resource allocation) are not solvable by iterated strict dominance
- A broad class of games are characterized by the Nash equilibrium solution.
- Nash equilibrium is a profile of strategies such that each player strategy is an optimal response to the other players' strategies
- Definition: A mixed strategy profile σ^* is a Nash equilibrium, if for all players i,

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right), \quad \forall\, s_i \in S$$

- Similar def. for the pure strategies

Note: If a player uses a nondegenerate strategy in a Nash eq., he must be Indifferent between all the pure strategies to which he assigns positive probability

Strict Nash equilibrium

- NE is strict if each player has a unique best response to his rivals' strategies
  - No mixed strategies
- s* is a strict Nash equilibrium iff is a Nash equilibrium, and for all i, and all s_i

$$s_i \neq s_i^* \Rightarrow u_i\left(s_i^*, s_{-i}^*\right) > u_i\left(s_i, s_{-i}^*\right)$$

Nash equilibrium properties

- Existence
  - Pure strategy Nash equilibrium may not exist
- Uniqueness
  - Nash equilibrium need not to be unique
- Efficiency
  - Pareto Optimality?

Example Games: Matching pennies

- 2 players – simultaneously announce heads or tails
- If match – player 1 wins, ow. player 2 wins

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Equilibrium: randomize with prob 1/2

If player 2, plays (1/2, ½), why not player 1 play H all the time?

How to compute randomized strategy for the matching pennies game?

- Indiference principle:
- Assume player 1 plays H with *p*; player 2 plays H with *q*
  - Player 1 → indiferent between its choices → value of *q*

$$q-(1-q)=-q+(1-q)\Rightarrow q=\frac{1}{2}$$

  - Player 2 → indiferent between its choices → value of *p*

$$p-(1-p)=-p+(1-p)\Rightarrow p=\frac{1}{2}$$

Example Game: Battle of sexes

Player 1 favors B; player 2 favors F

	B	F
B	1,2	0,0
F	0,0	2,1

An additional mixed strategy equilibrium:

- impose the two choices to be the same for the users
- denote x = probability that player 1 plays F
- denote y = probability that player 2 plays B

Player 1 indif: $0*y+2*(1-y)=1*y+0*(1-y) \Rightarrow y=2/3$

Player 2 indif: $2*(1-x)+0*x=0*x+2*(1-x) \Rightarrow x=2/3$

Game: Friend or Foe

- **Game:** At the end of a game show, the winning team may win 1000$. To win the money, the 2 members of the team must go in separate rooms and vote Friend or Foe. It is assumed that the two team mates are complete strangers. If both choose Friend, then the two players split the money equally. If one chooses Foe, while the other chooses Friend, then the former takes all the money. If both choose Foe, then both players lose, and they get nothing.

Friend or Foe: Nash equilibrium

	Friend	Foe
Friend	500,500	0,1000
Foe	1000,0	0,0

How to get to the Nash Equilibrium?

- Rational Introspection
  - Based on knowledge of the game → selects best response
- Focal Point
  - Best response for all players
- Trial and Error
  - Iterative procedure, to discover the equilibrium point by deviating from current action to improve payoff
- Pre-play communication
  - Only Nash equilibrium points can be negotiated, ow. players have incentives to deviate

Which equilibrium is the best?

Pareto efficiency:

An strategy profile is *Pareto optimal* if some players must be hurt in order to improve the payoff of other players

Def: A strategy profile s^* is said to be *Pareto optimal* iff there exists no other strategy profile s', such that if for some j

$$u_j(s') > u_j(s^*), \quad u_i(s') \geq u_i(s^*), \ \forall i \in I \setminus j$$

Observations:

A strategy profile that is a Nash equilibrium may not necessarily be Pareto optimal (efficient).
A strategy profile which is Pareto efficient, is not necessarily a Nash equilibrium.
We would like Nash equilibrium to be Pareto efficient.

Example Game

	a_1	a_2
a_1	2,3	-2,7
a_2	6,-5	0,-1

	a_1	a_2
a_1	2,3	-2,7
a_2	6,-5	3,5

Pareto efficient

One last example

- For the scissors, paper, rock game, the game matrix is depicted below

	scissors	paper	rock
scissors	0,0	1,-1	-1,1
paper	-1,1	0,0	1,-1
rock	1,-1	-1,1	0,0

- Please find the Nash equilibrium solutions for this game.