# CPE 345: Modeling and Simulation

Lecture 12

# Today’s topics

- Chapter 10: Output analysis
- Some announcements
  - Next lecture – final projects
  - Reminder: complete the end of semester assessment surveys

# Output analysis

- So we did all the work
  - Objective definition
  - Modeling
  - Validation
  - Computer implementation
  - Verification
  - We got results!  Now what?
- How do we interpret the outputs (results)?
- Why do we need to interpret results?
  - To predict the system performance
  - Compare performance of two or more design solutions

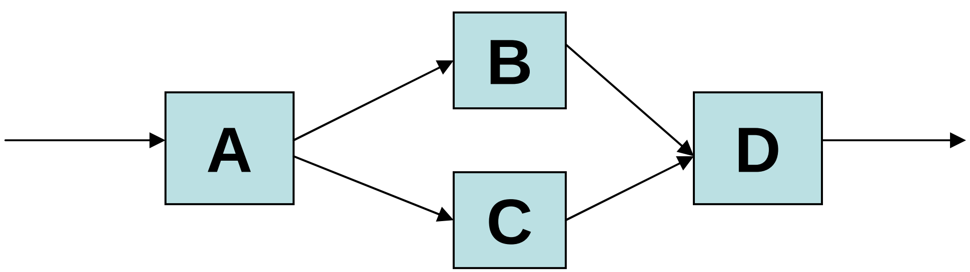
# How do we interpret the outputs (results)?

- Outputs – stochastic character → random processes
  - They may depend on the initial conditions
  - Sometimes are correlated
  - How to estimate your simulation performance?
  - What confidence do you have in your estimation?
    - How long you should run your simulation for a desired precision?

- With respect to the output analysis, simulations
  - Terminating (transient)
    - Simulation that runs for some duration of time $T_E$ (E is a specified event or set of events that stops the simulation)
  - Steady-state

# Example of terminating systems

- Example 11.1 → terminating time known in advance
  - Shady Grove bank – one day operation simulation
    - Opens 8.30 am, closes 4.30 pm
    - Initial conditions: 0 customers, 8 out of 11 tellers working
    - Simulation time 480 minutes
- Example 11.3 → terminating time not known in advance
  - Communication system with A, B, C, D components




  - Simulation stops when system fails {A fails or D fails, or (B and C fails)}
  - One possible performance measure: the mean time to system failure: $E(T_E)$

# Stochastic outputs

- Remember the two teller bank example in lecture 2
  - Simulation run for 20 arrivals of customers
    - Utilization for first server = 0.814
    - Average delay = 2.0
  - Suppose now that we run the simulation independently 4 times (using different seeds for the random generators), each simulation duration is 2 hours.

| run $r$ | Teller 1 utilization $\hat{\rho}_r$ | Delay $\hat{w}_r$ |
|---|---|---|
| 1 | 0.815 | 2.1 |
| 2 | 0.798 | 1.94 |
| 3 | 0.863 | 2.31 |
| 4 | 0.721 | 1.87 |

# Stochastic outputs – cont.

- Two questions need to be addressed:
  - Estimation of the true utilization – point estimate

$$\rho = E(\hat{\rho}_r)$$

  - Estimation of the error in our point estimate
    - Standard error
    - Confidence interval

# Measures of performance and their estimation

- System performance measure: $\theta$
- Estimator of $\theta = \hat{\theta}$
- Measure of precision of $\hat{\theta}$ = $\mathrm{var}(\hat{\theta})$

- Point estimation – based on data $\{Y_1, Y_2, ..., Y_n\}$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$ ← Sample mean on a sample of size *n*

"observational statistic"

The point estimator is unbiased for $\theta$ if $E(\hat{\theta}) = \theta$

In general, $E(\hat{\theta}) \neq \theta$ → $E(\hat{\theta}) - \theta$ is called the bias in the point estimator $\hat{\theta}$

# Point Estimation – continuous time

- Point estimator for $\varphi$ based on the data $\{Y(t),\ 0 \le t \le T_E\}$

$$\hat{\varphi} = \frac{1}{T_E}\int_0^{T_E} Y(t)dt$$ ⟵ Time average of *Y(t)* over [0, $T_E$]

"time-persistent statistic"

In general, $E(\hat{\varphi}) \neq \varphi$ → $\hat{\varphi}$ is said to be biased for $\varphi$

Both $\hat{\theta}$ and $\hat{\varphi}$ are mean measures for performance

Examples: recall the definitions for $\hat{L}$ and $\hat{w}$ for the queueing results:

$$\hat{L} = \frac{1}{T}\int_0^{T} L(t)dt \to L, \text{as } T \to \infty$$

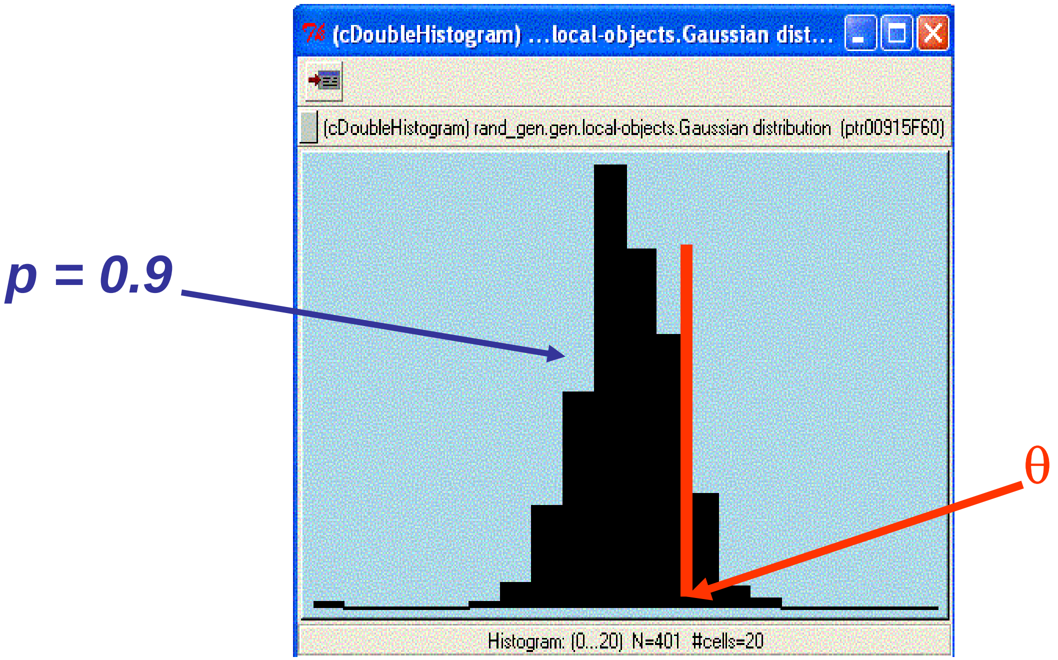
$$\hat{w} = \frac{1}{N}\sum_{i=0}^{\infty} W_i \to w, \text{as } T \to \infty, \text{N} \to \infty$$

# Quantile estimation

- Quantile = describes the level of performance that can be given with a given probability *p*
  - Example: statistical guarantees in a network:
    - End-to-end delay for a packet < 250 ms with probability $p = 0.9$
    - The 0.9 quantile of end-to-end delay is 250 ms
- Different approach: we know *p*, we have to determine $\theta$
  - *Hint: use histograms*

# Quantile estimation- solution

– *Build a histogram → select $\theta$ s.t. 100*p% of the histogram is to the left of $\theta$*



# Interval estimation

- Performance value may vary significantly around the point estimator
- We need to estimate the variance of our estimator for the given number of samples → we can determine how many samples we need for a given desired variance
- The true variance of the point estimator:

$$\sigma^2\left(\hat{\theta}\right) = \mathrm{var}\left(\hat{\theta}\right)$$

- The estimate of the variance of the point estimator is

$$\hat{\sigma}^2\left(\hat{\theta}\right)$$

  – Which is generally biased

# Confidence Intervals

- If the variance estimator for the point estimator is approximately unbiased, then the statistic

$$t = \frac{\hat{\theta} - \theta}{\hat{\sigma}\left(\hat{\theta}\right)}$$

- is approximately *t*-distributed with *f* degrees of freedom (f = n-1, where n = number of samples)
  - An approximate 100(1- $\alpha$)% confidence interval for $\theta$ is given by

$$\hat{\theta} \pm t_{\alpha/2,f}\,\hat{\sigma}\left(\hat{\theta}\right)$$

$$P\left(t \geq t_{\alpha/2,f}\right) = \alpha/2$$

  - The difficulty is to obtain unbiased estimates for the variance from the simulation

# Determine confidence intervals

- Suppose $\{Y_1, Y_2, ..., Y_n\}$ are statistically independent observations
  - Unbiased estimators for mean and variance

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$S^2 = \sum_{i=1}^{n} \frac{(Y_i - \theta)^2}{n-1} = \frac{\sum_{i=1}^{n} Y_i^2 - n\hat{\theta}^2}{n-1}$$

Unbiased estimator for the population variance: $\sigma^2 = \mathrm{var}(Y_i)$

# Determine confidence intervals – cont.

- The variance for $\hat{\theta}$ is given by

$$\sigma^2\left(\hat{\theta}\right) = \frac{\sigma^2}{n}$$

- An unbiased estimator for $\sigma^2(\hat{\theta})$ is $\hat{\sigma}^2(\hat{\theta}) = \frac{S^2}{n}$
- The quantity $\hat{\sigma}(\hat{\theta}) = \frac{S}{\sqrt{n}}$ is called the standard error of the point estimator $\hat{\theta}$

  → measure of the precision of a point estimator = average deviation to be expected between the point estimator and the true mean

Since $\hat{\sigma}^2(\hat{\theta})$ is unbiased, the confidence interval can be determined as before

# Output analysis for terminating simulations

- Simulation runs over a time interval $[0, T_E]$
- Output observations: $\{Y_1, Y_2, ..., Y_n\}$
- We want to estimate $\theta$

$$\theta = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)$$

- Repeat simulation R times, with random initial conditions → independent number streams from run to run
  - If $Y_{ri}$ = the i-th observation within replication r

- The sample mean can be defined as:

$$\hat{\theta}_r = \frac{1}{n_r}\sum_{i=1}^{n_r} Y_{ri}, \quad r = 1, ..., R$$

Method of independent replications

# Terminating simulations: confidence interval

- The variance estimator is given as

$$\hat{\sigma}^2\left(\hat{\theta}\right)=\frac{1}{(R-1)R}\sum_{r=1}^{R}\left(\hat{\theta}_r-\hat{\theta}\right)^2$$

- A 100(1-$\alpha$)% confidence interval is given by

$$\hat{\theta}-t_{\alpha/2,f}\,\hat{\sigma}\left(\hat{\theta}\right)\le\hat{\theta}\le\hat{\theta}+t_{\alpha/2,f}\,\hat{\sigma}\left(\hat{\theta}\right)$$

  – With f = R-1

# The bank tellers example revisited

| run | Teller 1 utilization $\hat{\rho}_r$ | Delay $\hat{w}_r$ |
|---|---|---|
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- Determine $\hat{\rho} = \dfrac{0.815 + 0.798 + 0.863 + 0.721}{4} = 0.79925$
- Compute

$$\hat{\sigma}^2(\rho) = \frac{(0.815 - 0.79925)^2 + ... + (0.721 - 0.79925)^2}{3 * 4} = 0.001$$

# Bank tellers example -cont

- The standard error is determined as $\hat{\sigma}\left(\hat{\rho}\right) = 0.032$

- For a 95% confidence interval → 100(1-$\alpha$)% = 95 → $\alpha$ = 0.05 → $\alpha$/2 = 0.025

- From table at the end of the book, the 95% confidence interval is

$$0.697 \leq \rho \leq 0.901$$

where $t_{0.025,3=3.18}$

# Output analysis for steady-state simulation

- A single run of a simulation model
- Goal: estimate a steady-state parameter
- Output: $Y_1$, $Y_2$, …, → samples of a correlated time series
- The estimate:

$$\theta = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_i$$

- Simulation cannot run forever: how to choose n to terminate simulation?
  - Eliminate bias in estimators due to inappropriate initial conditions → significant for short simulations
  - Desired precision/confidence interval

# Homework

- No homework due for next class
- Next class – project presentations