Homework 5 solution – due 10/14/03
Chapter 6, page 248, problem 11, page 251, problem 23

11. Classic Car Care has one worker who washes cars in a four-step method – soap, rinse, dry, vacuum. The time to complete each step is exponentially distributed with a mean of 9 minutes. Every car goes through every step before another car begins the process. On the average, one car every 45 minutes arrives for a wash job, according to a Poisson process. What is the average time a car waits to begin the wash job? What is the average number of cars in the car wash system? What is the average time required to wash a car?

Model the car wash as an M/G/1 queue with $\lambda=1/45$, $\mu=1/36$ (4 steps with each taking 9 minutes means that the total time is 4*9=36 minutes, on average), The variance (σ^2) of the sum of 4 exponential distributions, each having a mean of 9 minutes is 324, since the variance of the sum of independent RVs is the sum of the variance – remember that each step has a mean of 9 minutes, so it's variance is 81 minutes^2. $\rho=\lambda/\mu=36/45 = .8$. Note: this is NOT an M/M/1 queue, since the sum of exponential service times is not exponential but, instead, is a Gamma distribution.

The average time a car waits to begin the wash job is:

$$w_Q = \frac{\lambda\left(\frac{1}{\mu^2}+\sigma^2\right)}{2(1-\rho)} = \frac{\frac{1}{45}\left(36^2+324\right)}{2(1-.8)} = 90 \text{ minutes}$$

The average number of cars in the car wash system is:

$$L = \rho + \frac{\rho^2(1+\sigma^2\mu^2)}{2\left(1-\rho\right)} = .8 + \frac{.8^2(1+324\cdot\frac{1}{36^2})}{2\left(1-.8\right)} = 2.8 \text{ cars}$$

The average time required to wash a car is

$$\frac{1}{\mu} = 36 \text{ minutes}$$

23. A small copy shop has a self-service copier. Currently, there is room for only 4 customers to line up for the machine (including the person using the machine); when there are more than 4 people, then the additional people must line up outside the shop. The owners would like to avoid having people line up outside the shop as much as possible. For that reason, they are thinking about adding a second self-service copier. Self-service customers have been observed to arrive at a rate of 24 per hour, and they use the machine 2 minutes, on average. Assess the impact of adding another copier. Carefully state any assumptions you make.

This system could be modeled as a M/M/c queue with $\lambda=24$ per hour and $\mu=1/2$ per minute = 30 per hour, so $\rho=24/30=.8$. Note: This is not a system with limited capacity, since the customers who cannot wait in the store *can* wait outside. We can compute the probability that customers have to wait outside as:

$$p = \sum_{n=5}^{\infty} P_n = 1 - \sum_{n=0}^{4} P_n$$

Unfortunately, we do not have an expression for Pn for a M/M/c queue. Alternatively, we can model the system as an M/M/1 queue. In the case of two copiers, we can model this as a single copier with double the service rate, μ=60 per hour, ρ=.4.

In this case,

$$P_n = (1 - r)\, r^n$$

$$p = 1 - \sum_{n=0}^{4} (1 - r)\, r^n$$

For the single copier case,

$$p = 1 - \sum_{n=0}^{4} (1 - .8).8^n = .328$$

For the two copier case,

$$p = 1 - \sum_{n=0}^{4} (1 - .4).4^n = .01$$

In other words, with only one copier, there is a 33% chance that customers will have to wait outside. By introducing a second copier, the probability that a customer will have to wait outside is reduced to 1%.