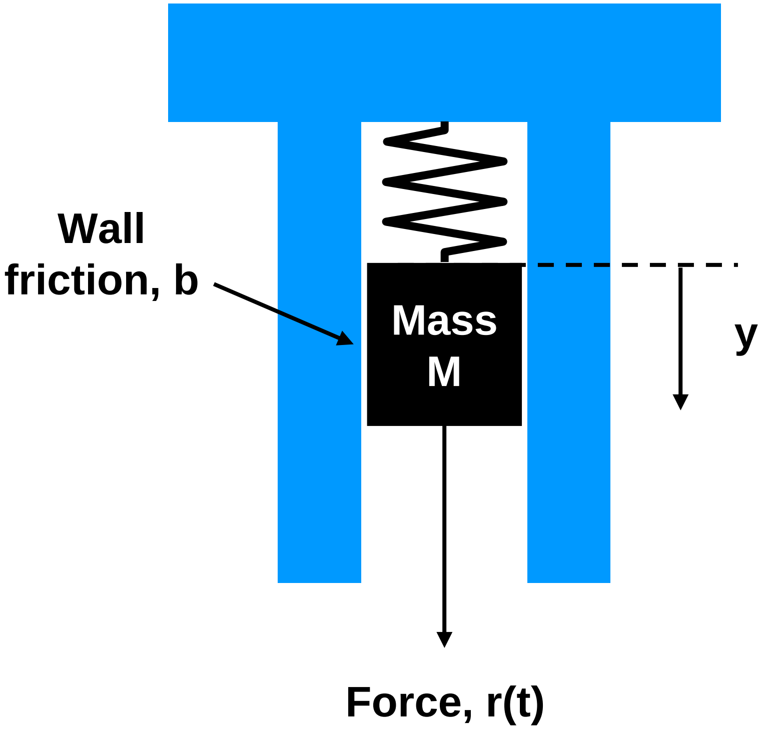
Design IV

E232 Spring 07

Class 24

Bruce McNair
bmcnair@stevens.edu

Analyzing Dynamic Systems – Differential Equations



$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

Using the Laplace Transform to Replace Differential Equations

- Simplify:

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

$$M\left(s^2Y(s)-sy(0-)-\frac{dy(0-)}{dt}\right)+b\left(sY(s)-y(0-)\right)+kY(s)=R(s)$$

$$r(t)=0 \qquad y(0-)=y_0 \qquad \left.\frac{dy}{dt}\right|_{t=0-}=0$$

$$Ms^2Y(s)-Msy_0+bsY(s)-by_0+kY(s)=0$$

Using the Laplace Transform to Solve Differential Equations

- Rearrange to solve for Y(s)

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

$$Y(s) = \frac{(Ms+b)y_0}{Ms^2+bs+k} = \frac{p(s)}{q(s)}$$

Determines "characteristic equation"

- Example, let k/M=5, b/M=6

$$Y(s) = \frac{(s+\frac{b}{M})y_0}{s^2+\frac{b}{M}s+\frac{k}{M}} = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{p(s)}{q(s)}$$

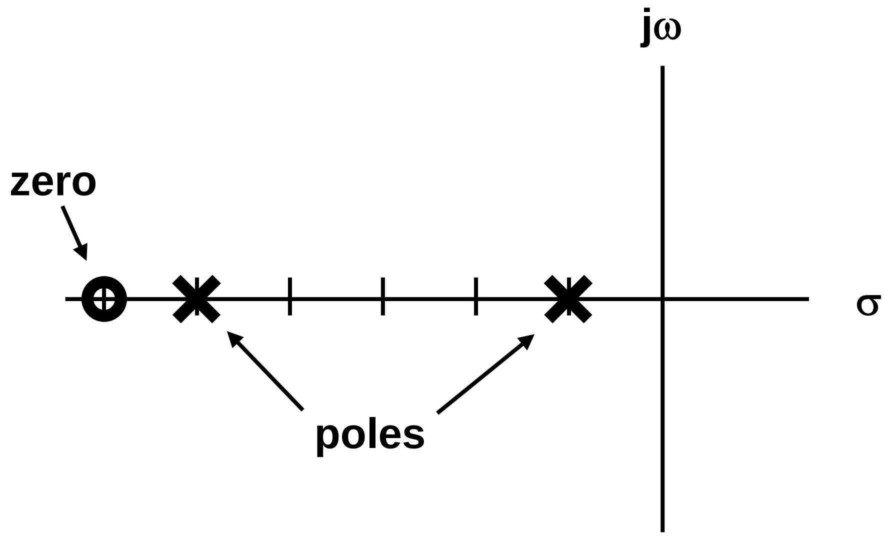
Using the Laplace Transform to Solve Differential Equations

- "Poles" and "zeroes" of Y(s)

At s=-6, Y(s)=0

$$Y(s) = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{p(s)}{q(s)}$$

At s=-1 or s=-5, Y(s) increases without bound



Using the Laplace Transform to Solve Differential Equations

- Partial fraction expansion of Y(s), assume y_0=1

$$Y(s) = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{k_1}{s+1} + \frac{k_2}{s+5}$$

$$k_1 = \left.\frac{(s-s_1)p(s)}{q(s)}\right|_{s=s_1} = \left.\frac{(s+1)(s+6)}{(s+5)(s+1)}\right|_{s=-1} = \frac{5}{4}$$

$$k_2 = \left.\frac{(s-s_2)p(s)}{q(s)}\right|_{s=s_2} = \left.\frac{(s+5)(s+6)}{(s+5)(s+1)}\right|_{s=-5} = \frac{1}{-4} = -\frac{1}{4}$$

Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$Y(s) = \frac{(s+6)}{(s+5)(s+1)} = \frac{5/4}{s+1} + \frac{-1/4}{s+5}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1} + \frac{-1/4}{s+5}\right\}$$

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1/4}{s+5}\right\}$$

Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1/4}{s+5}\right\}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$$

Using the Laplace Transform to Solve Differential Equations

- Final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

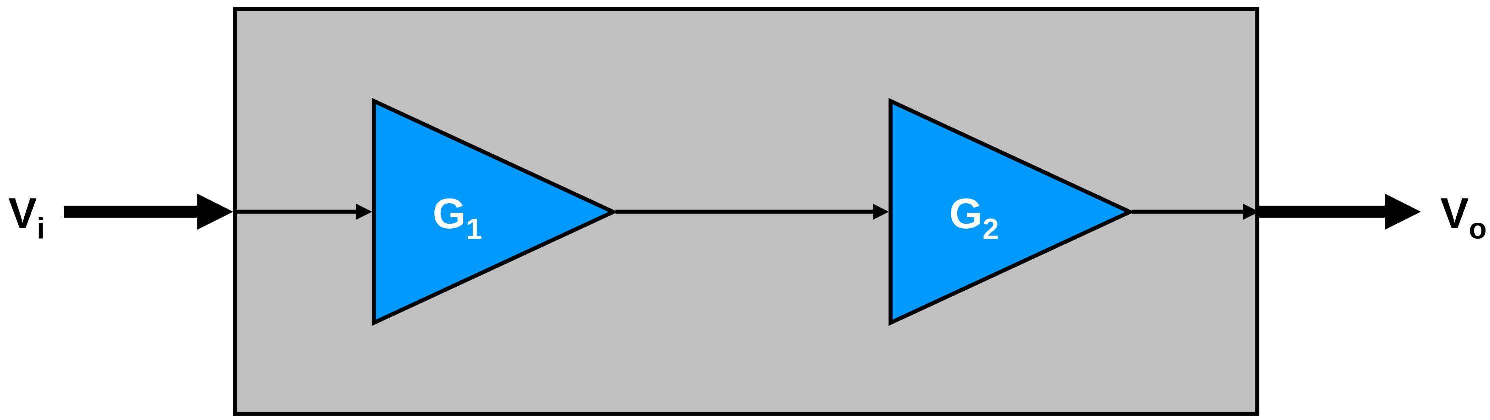
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s(s+6)}{(s+5)(s+1)} = \frac{0\cdot 6}{5\cdot 1} = 0$$

Today's topics

- Control systems
  - Transfer function
  - Steady state response of proportional control system

Representing Stages As Gain Blocks

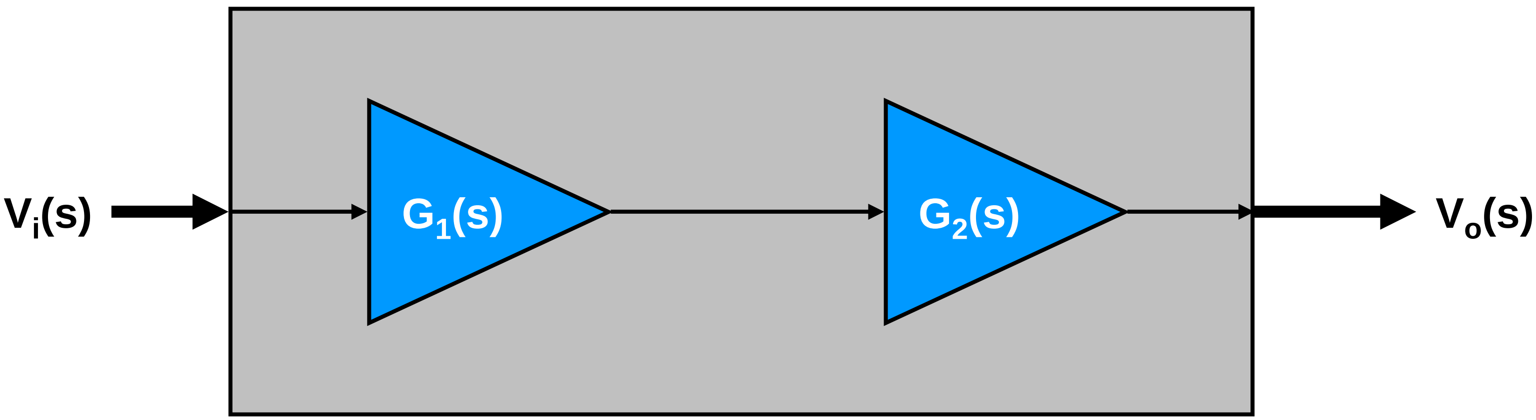
- Consider two amplifiers in cascade



$$G_{total} = \frac{V_o}{V_i} = G_1 G_2$$

Representing Stages As Transfer Function Blocks

- Generalize G to be a function of s

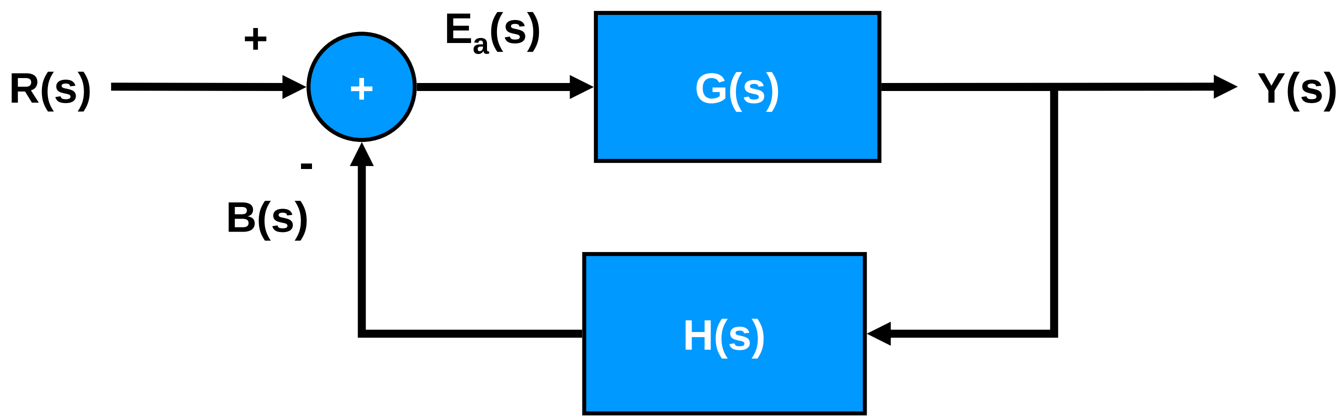


$$G_{total}(s) = \frac{V_o(s)}{V_i(s)} = G_1(s)G_2(s)$$

As long as the functions are linear

Transfer Function of a Feedback System

- Consider a generic feedback control system



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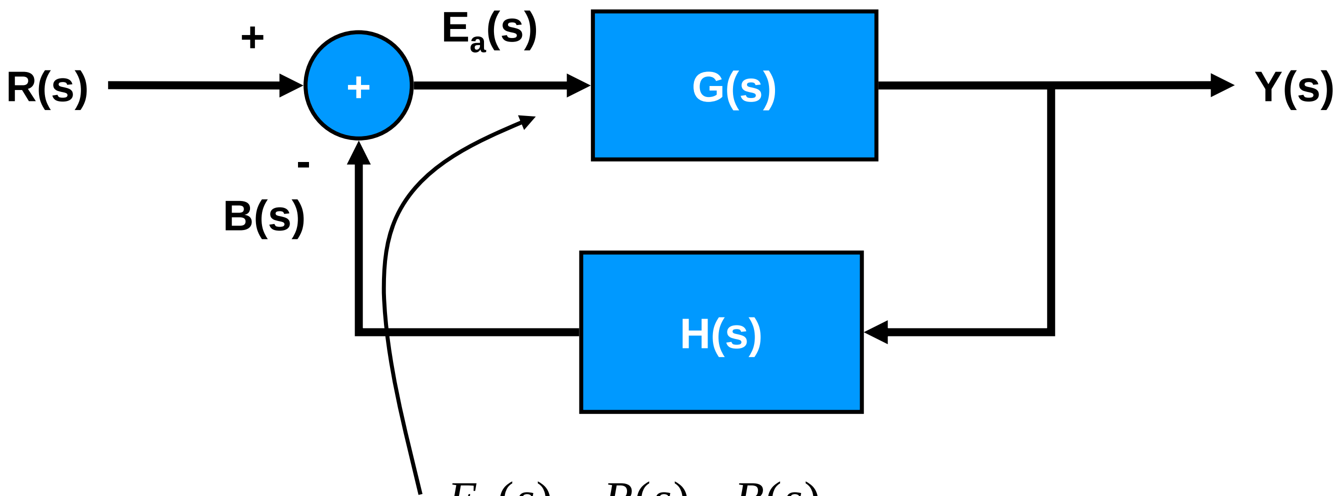

$$E_a(s) = R(s) - B(s)$$

Transfer Function of a Feedback System

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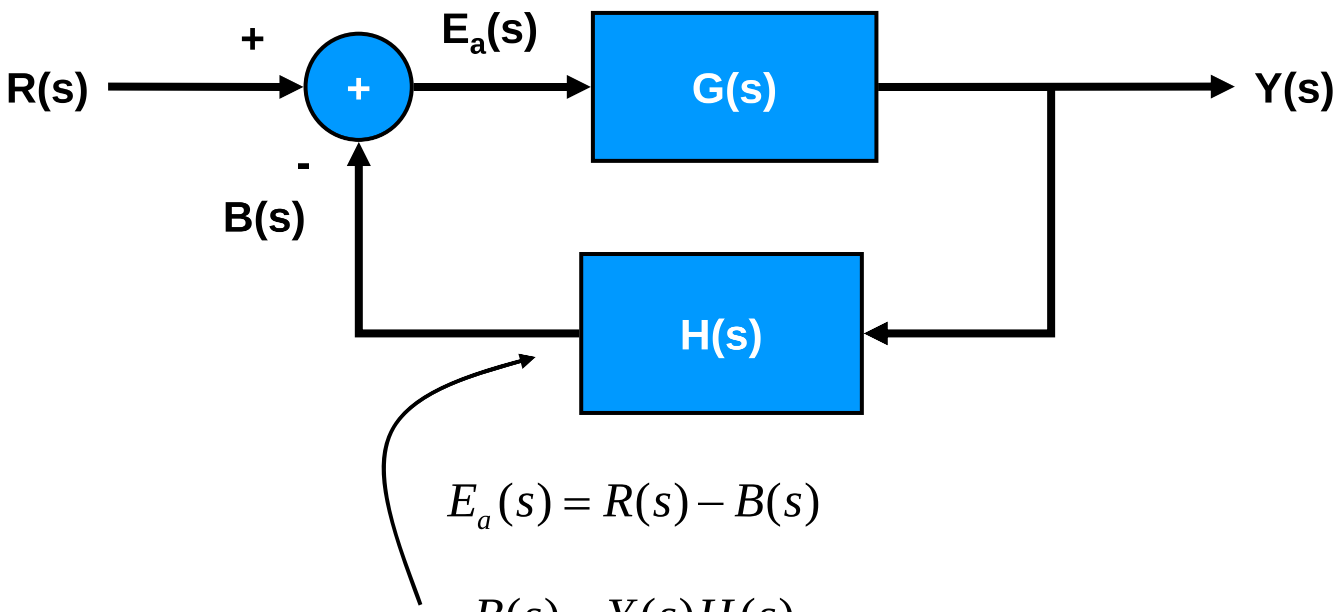
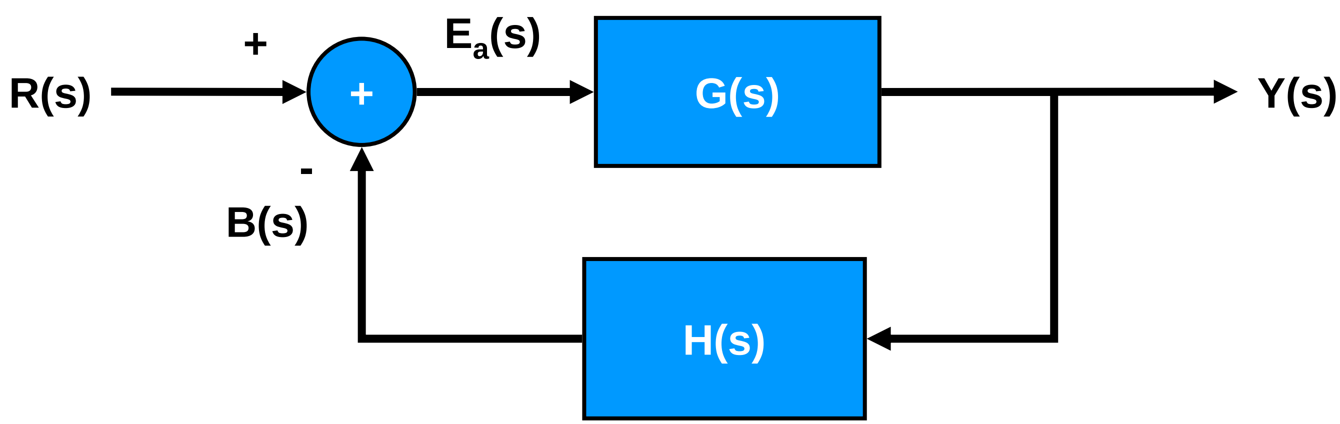


$$E_a(s) = R(s) - B(s)$$

$$B(s) = Y(s)H(s)$$

Transfer Function of a Feedback System

- Consider a generic feedback control system



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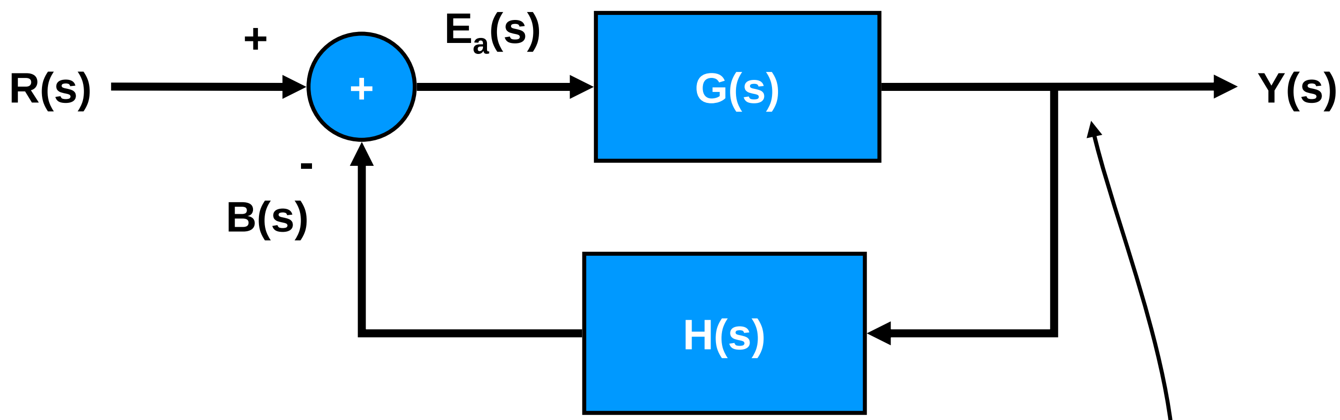

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$$Y(s) = E_a(s)G(s)$$

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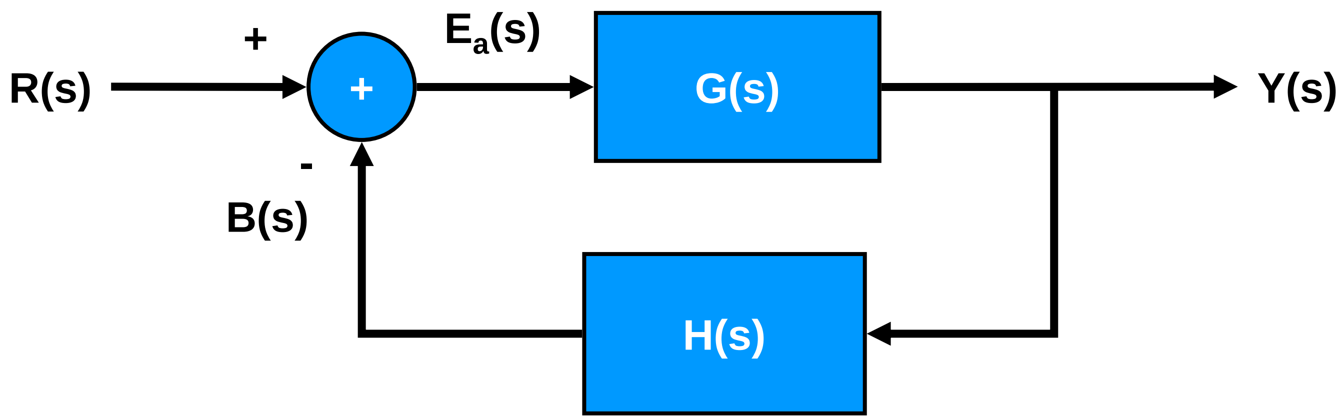

$$E_a(s) = R(s) - B(s)$$

$$Y(s) = \left(R(s) - Y(s)H(s)\right)G(s)$$

$$B(s) = Y(s)H(s)$$

$$E_a(s) = R(s) - Y(s)H(s)$$

$$Y(s) = E_a(s)G(s)$$

Transfer Function of a Feedback System

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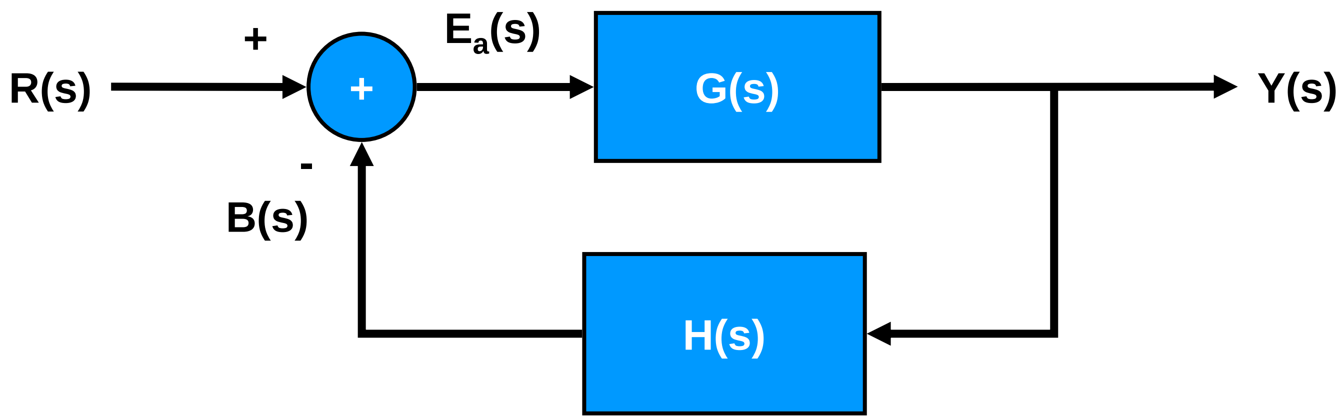

$$Y(s) = \left(R(s) - Y(s)H(s)\right)G(s)$$

$$Y(s)\left[1 + G(s)H(s)\right] = R(s)G(s)$$

Transfer Function of a Feedback System

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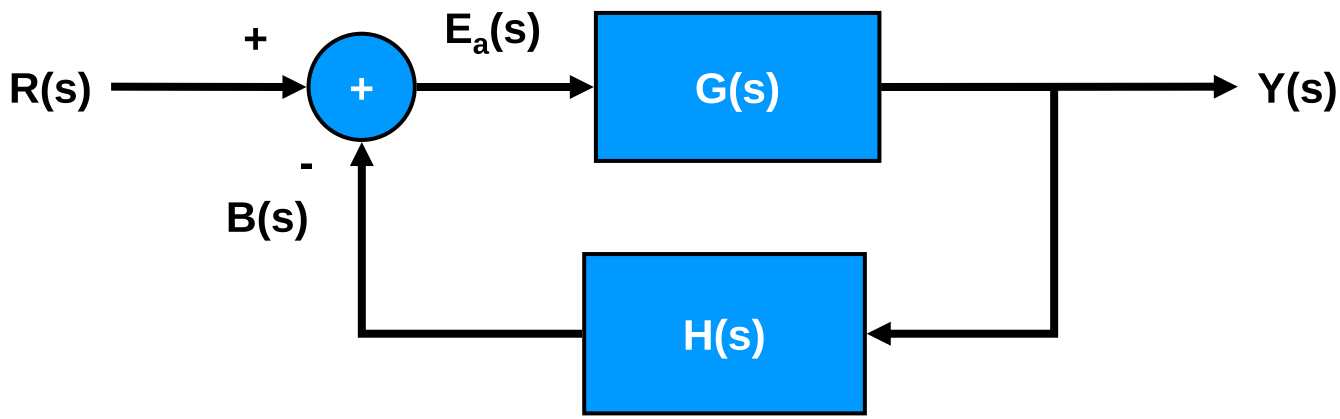

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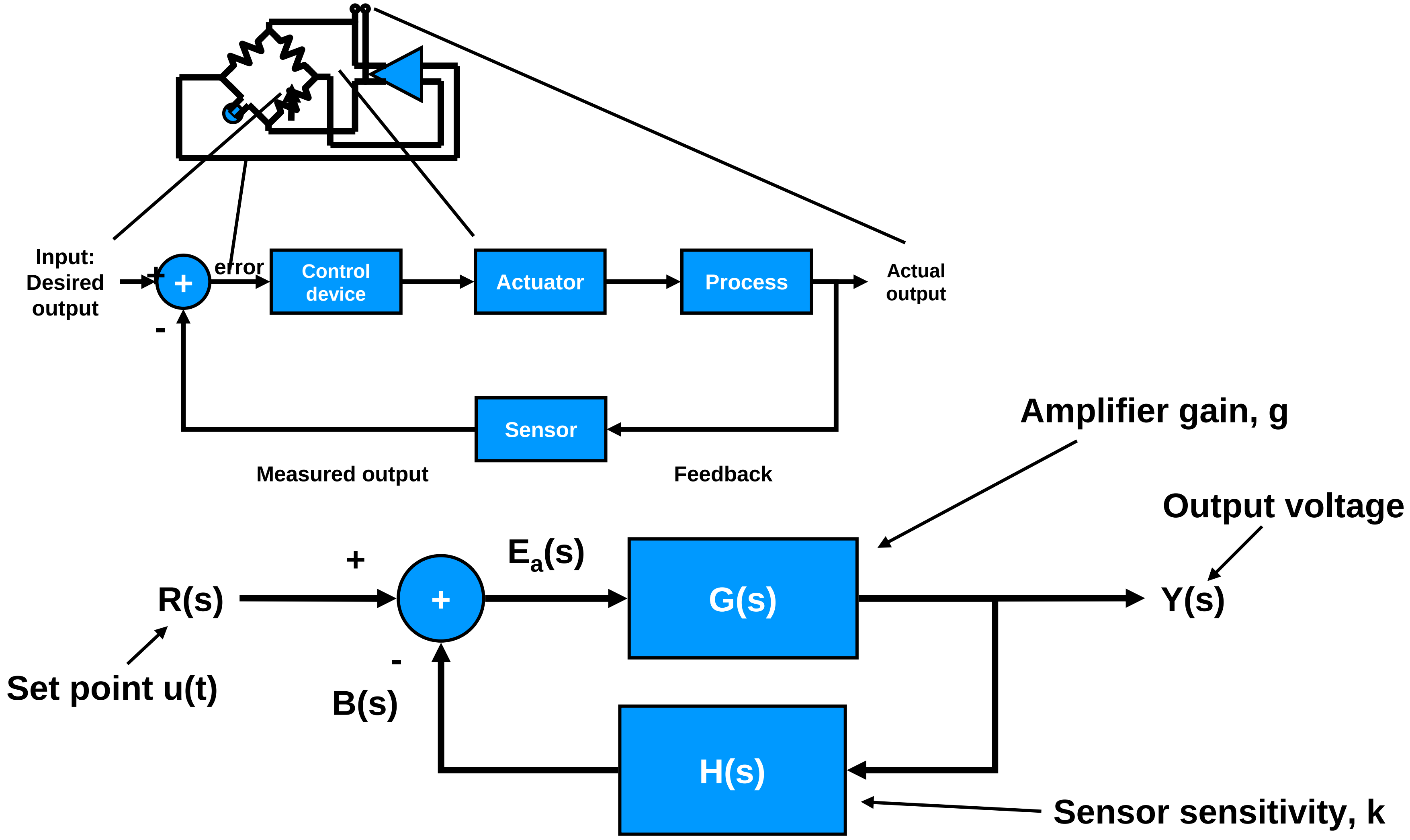
$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system



Steady State Response

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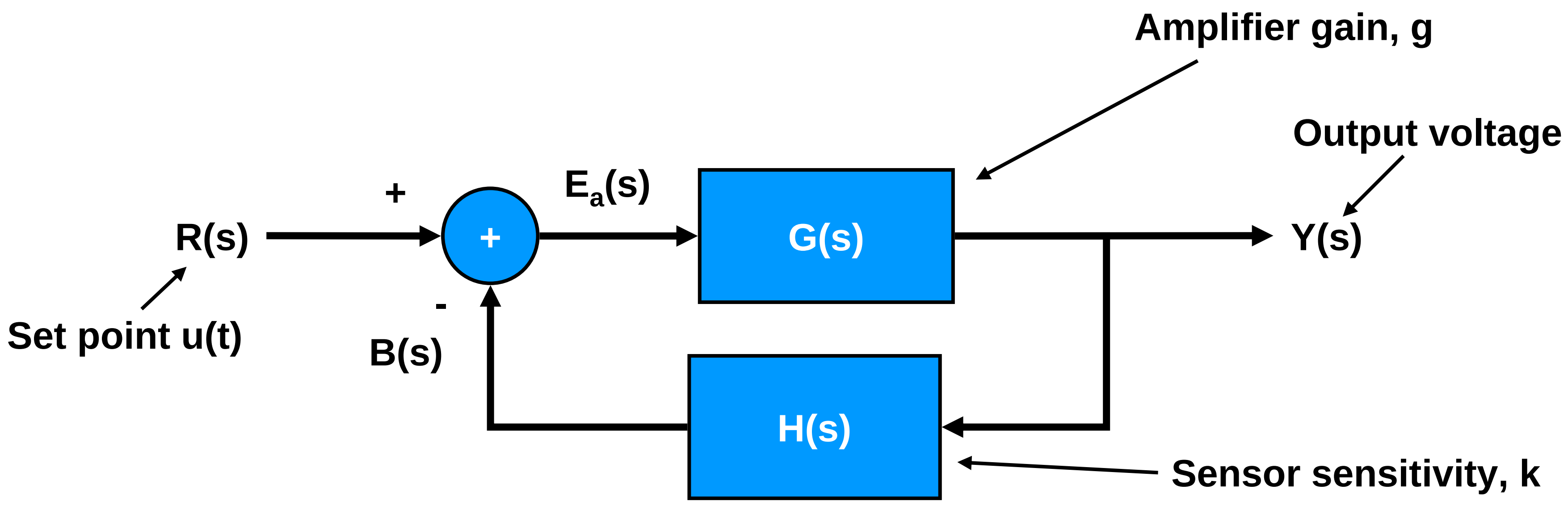

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$Y(s) = \frac{G(s)}{1+G(s)H(s)} R(s)$$

Steady State Response

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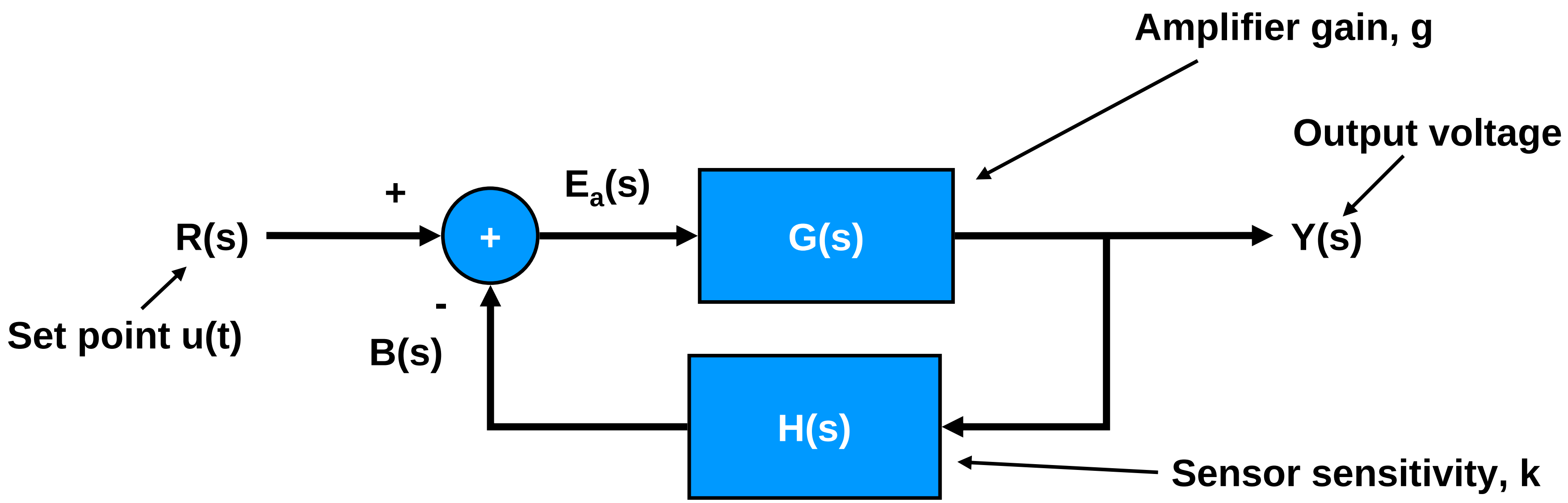


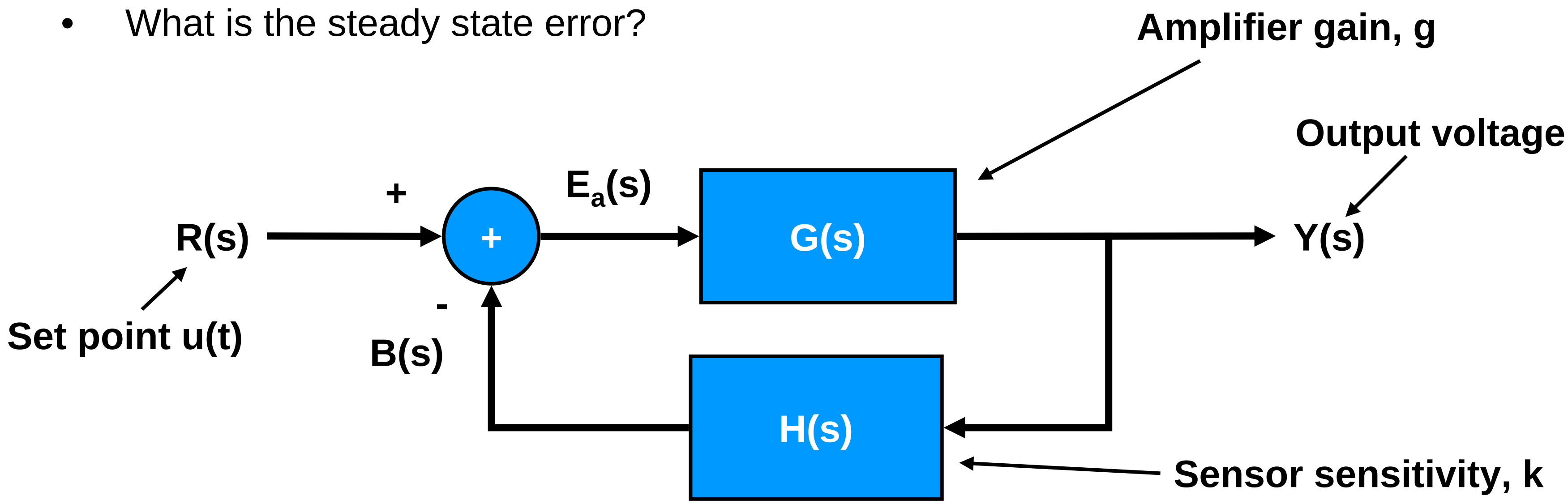
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$Y(s) = \frac{g}{1+gk}\frac{1}{s}$$

$$Y(s) = \frac{G(s)}{1+G(s)H(s)}R(s)$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state error?



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$$Y(s) = \frac{g}{1+gk}\frac{1}{s}$$

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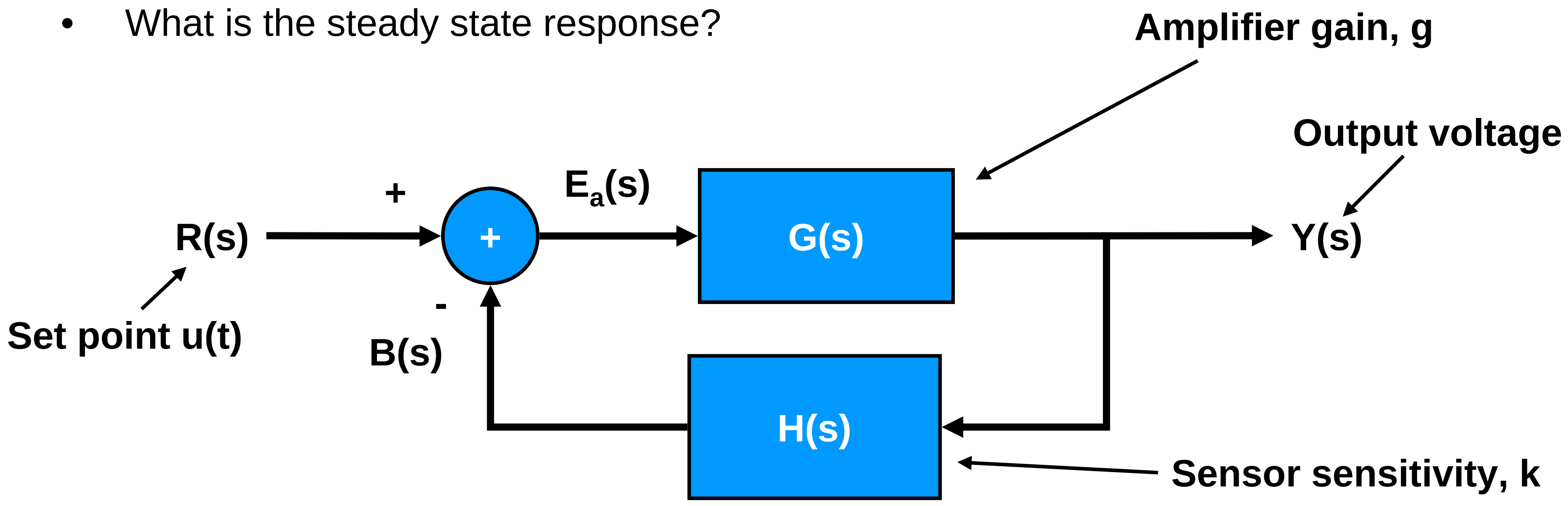
$$E_a(s) = \frac{g}{1+gk}\frac{1}{s}\frac{1}{g}$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?



$$E_a(s) = \frac{1}{s(1+gk)}$$

From the final value theorem:

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} sZ(s)$$

Steady State Response

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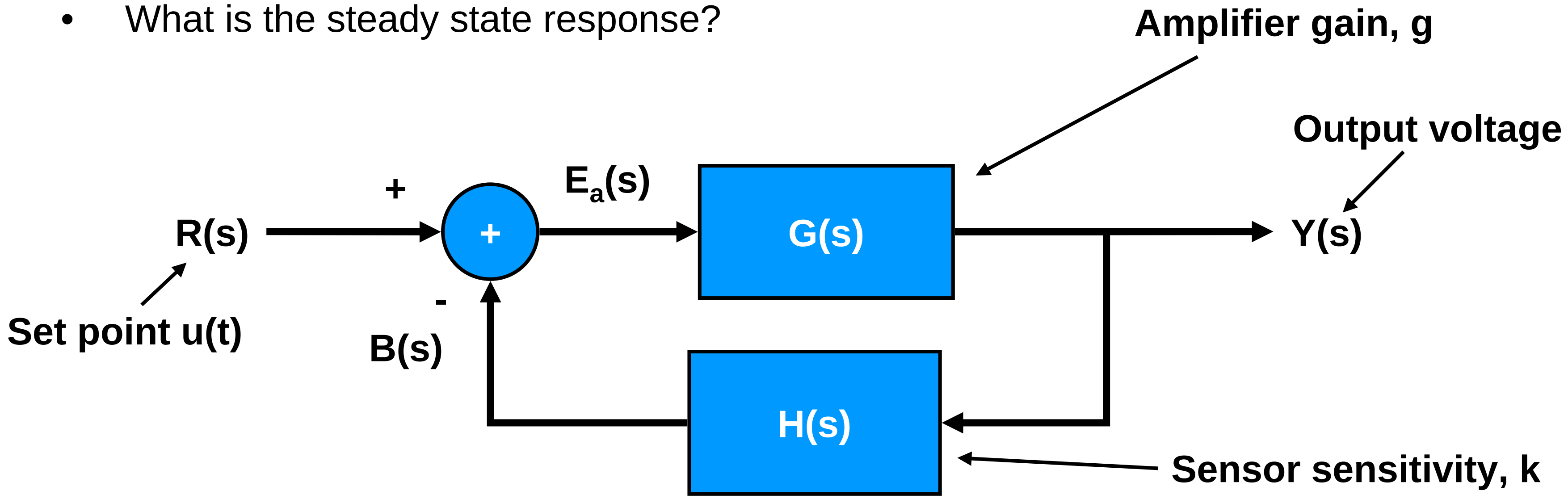

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Steady State Response

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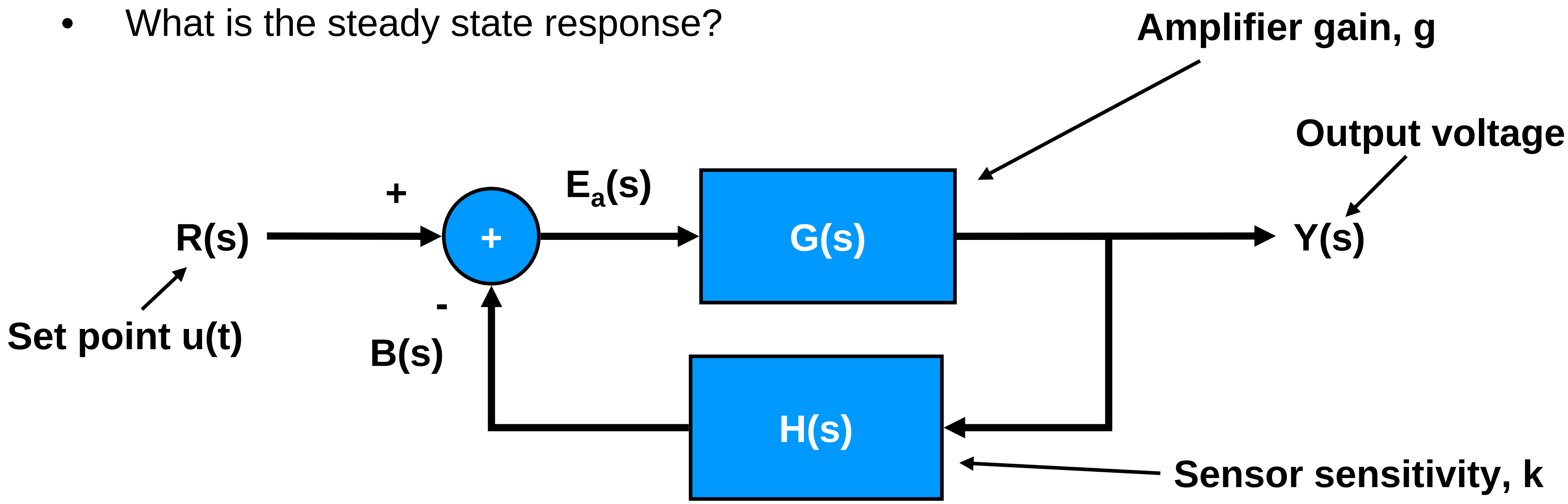

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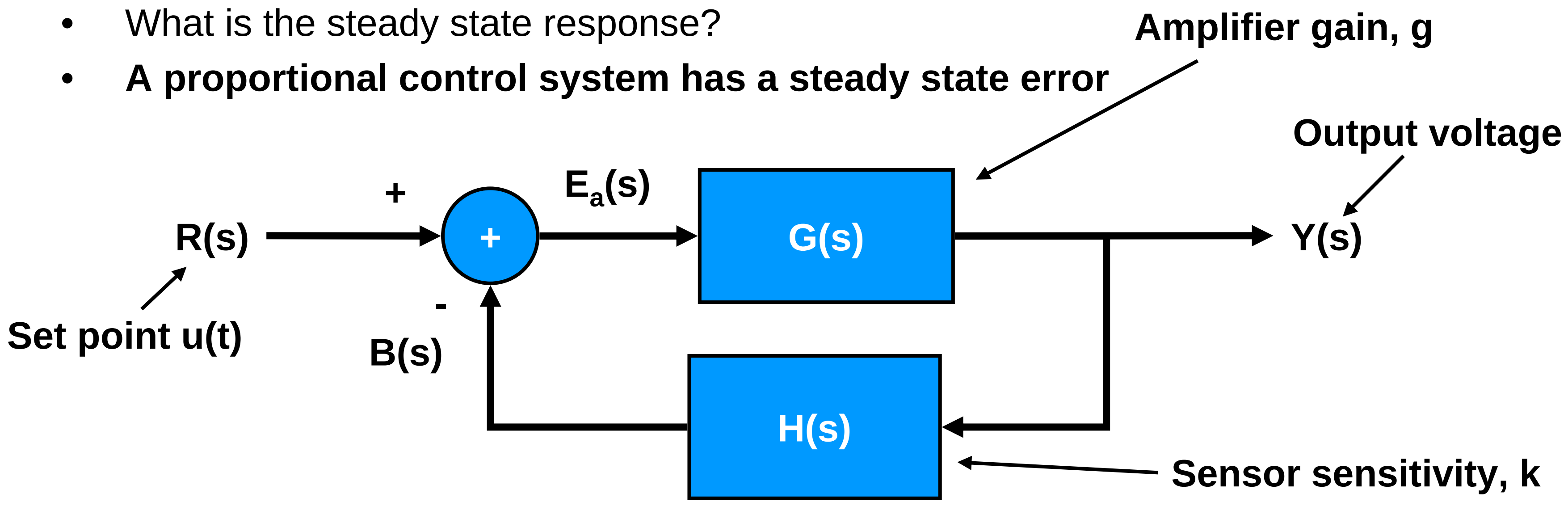
Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?
- **A proportional control system has a steady state error**



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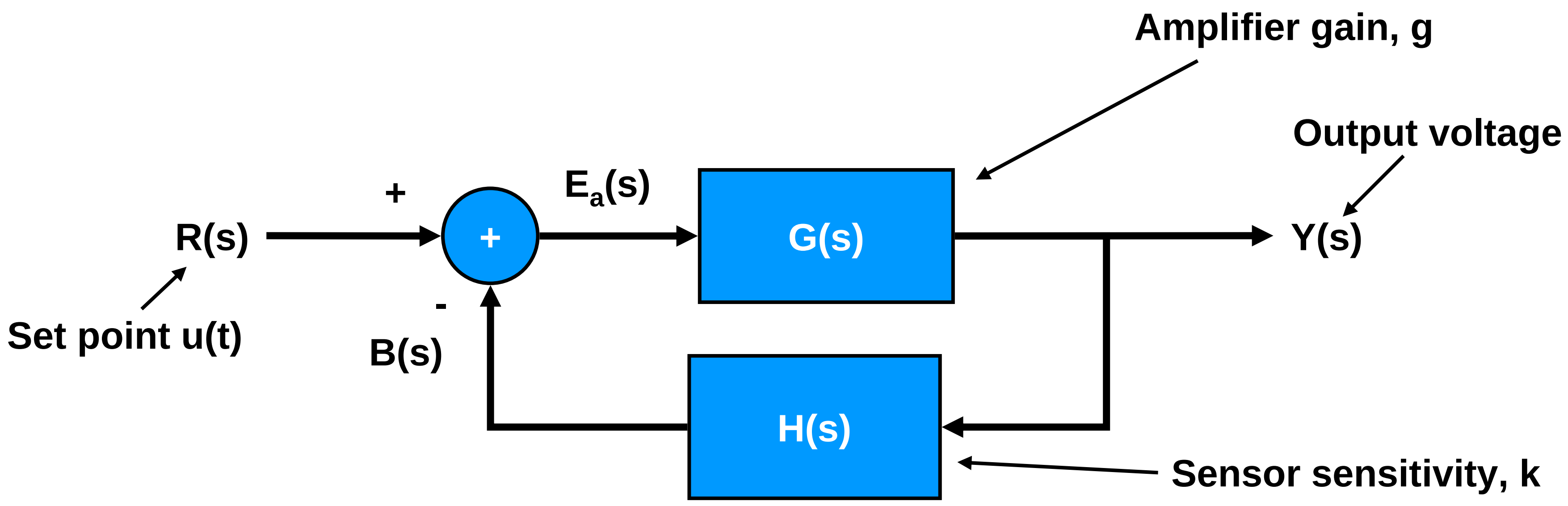
$$\boxed{\lim_{t\to\infty} e_a(t) = \frac{1}{1+gk}}$$

Steady State Response

- To reduce the steady state error:

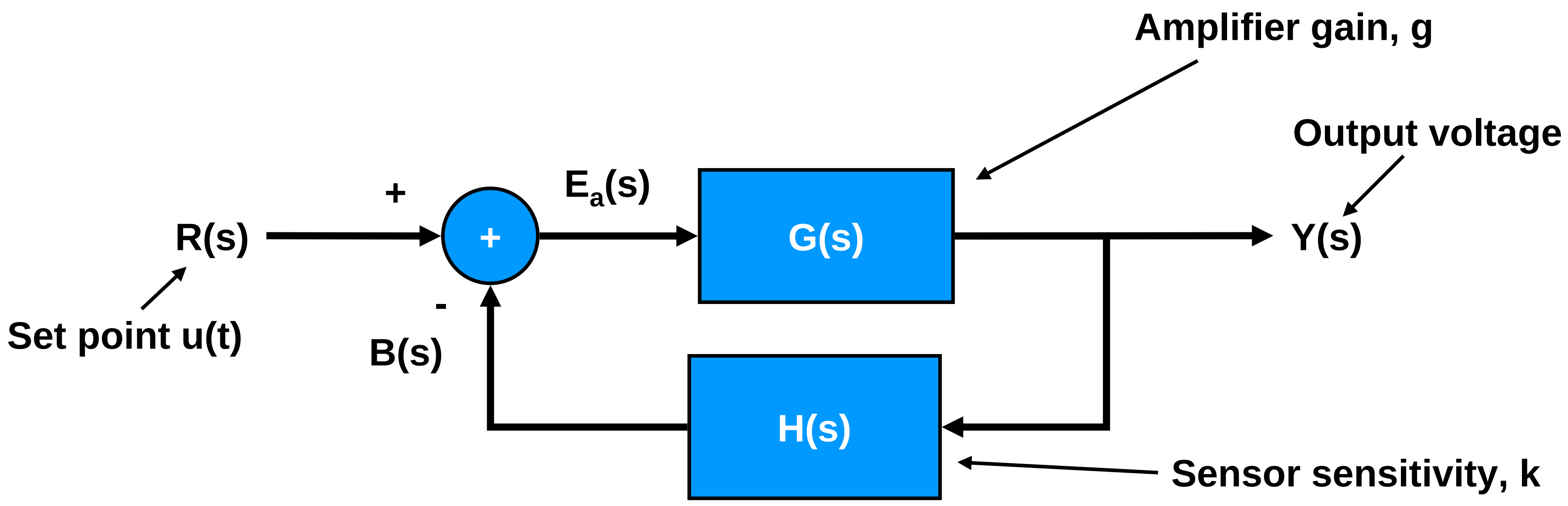


$$\lim_{t \to \infty} e_a(t) = \frac{1}{1+gk}$$

Increase gain, either in amplifier or sensor

Steady State Response

- To reduce the steady state error:



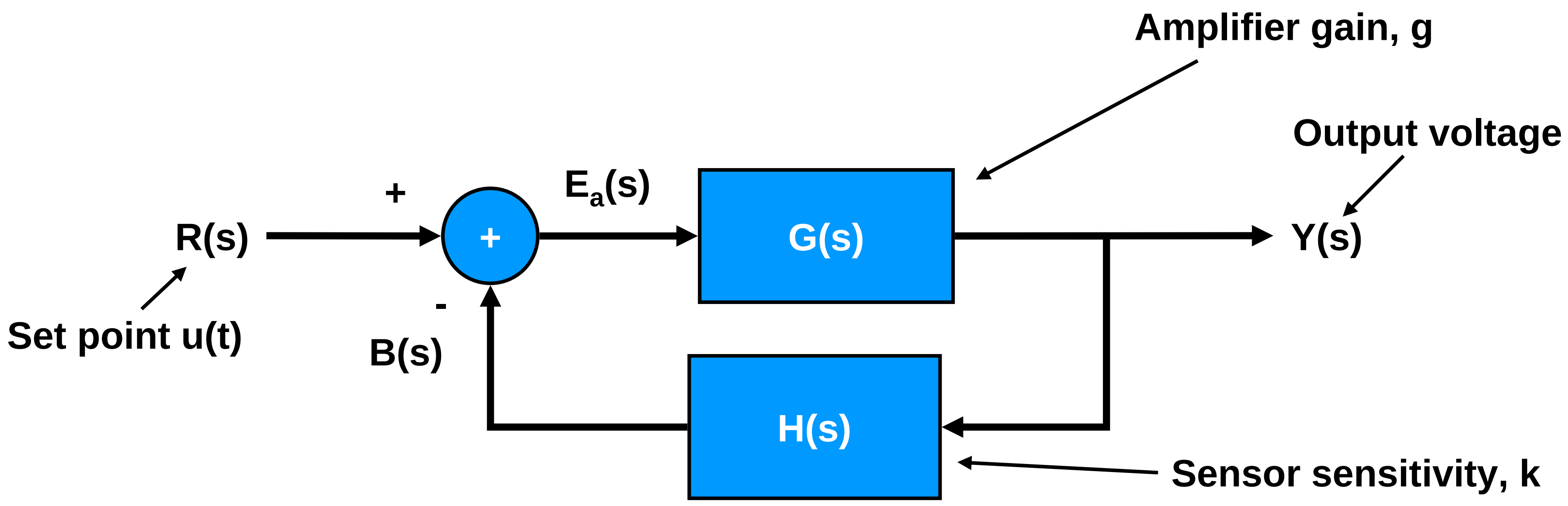
$$\lim_{t \to \infty} e_a(t) = \frac{1}{1 + gk}$$

Increase gain, either in amplifier or sensor

Making system more susceptible to noise!

Steady State Response

- To reduce the steady state error:



But, what if error signal was something like:

$$\lim_{t \to \infty} e_a(t) = \lim_{s \to 0} s \frac{1}{s\left(1 + g\frac{k}{s}\right)}$$

Steady State Response

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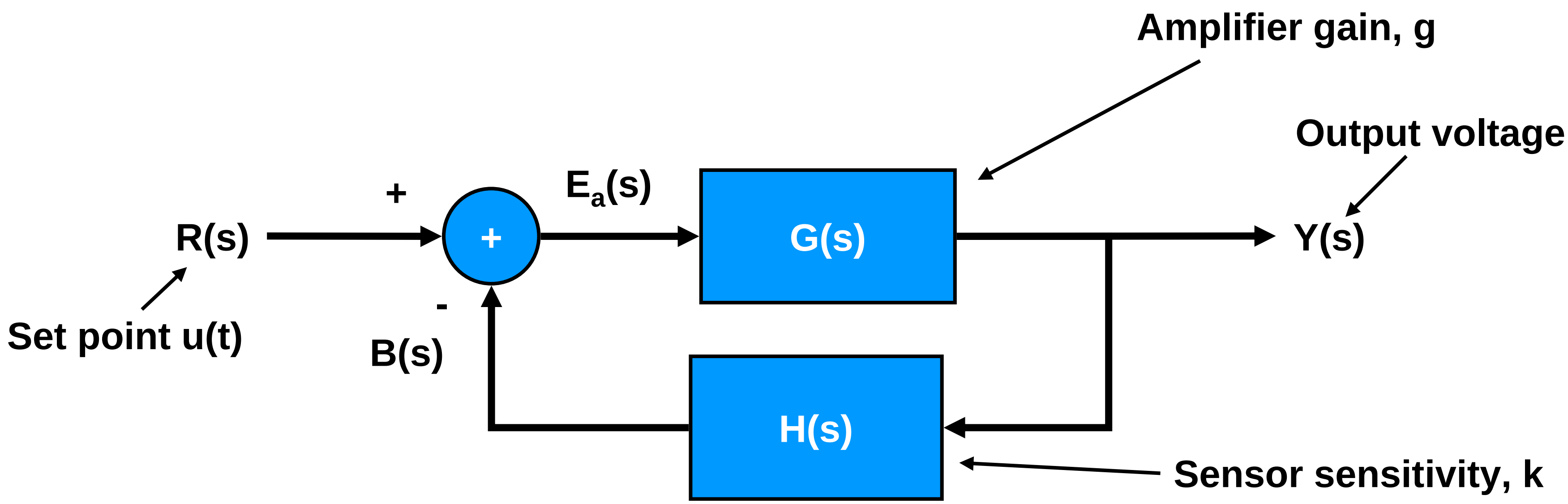

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Then:

$$\lim_{t \to \infty} e_a(t) = \lim_{s \to 0} \frac{1}{\left(1 + g\frac{k}{s}\right)} = \frac{s}{s + gk} = 0$$

Steady State Response

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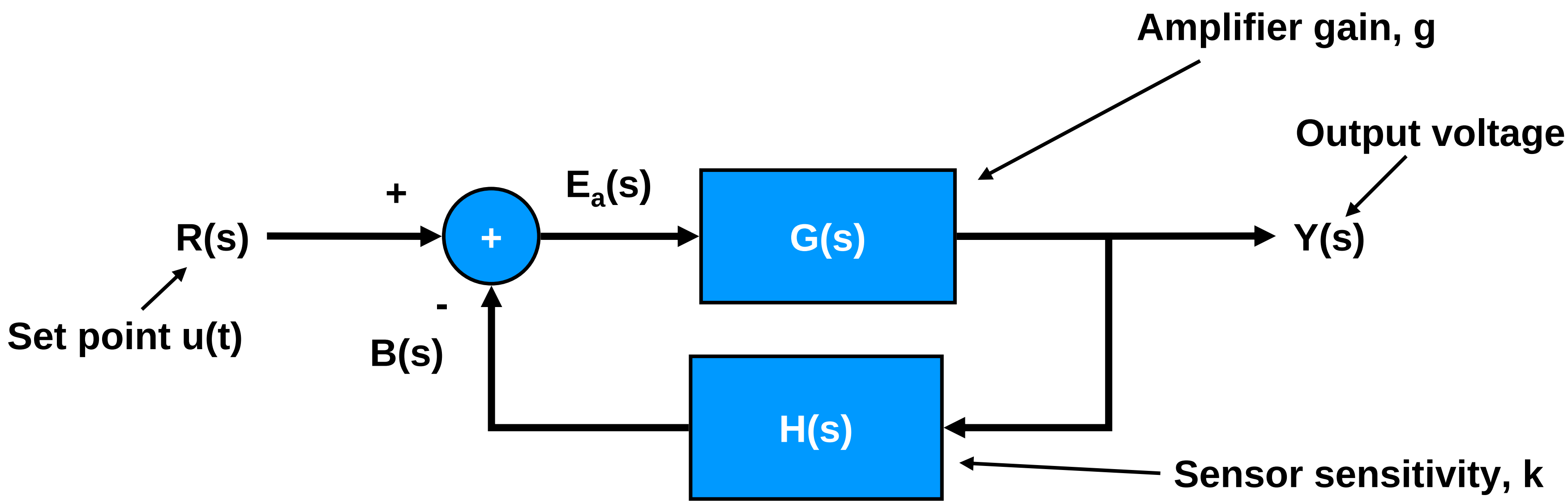

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$$\lim_{t\to\infty} e_a(t) = \lim_{s\to 0} s \frac{1}{s\left(1 + g\frac{k}{s}\right)}$$

Remember

$$\frac{1}{s} \equiv \int dt$$

Then:

$$\lim_{t\to\infty} e_a(t) = \lim_{s\to 0} \frac{1}{\left(1 + g\frac{k}{s}\right)} = \frac{s}{s + gk} = 0$$

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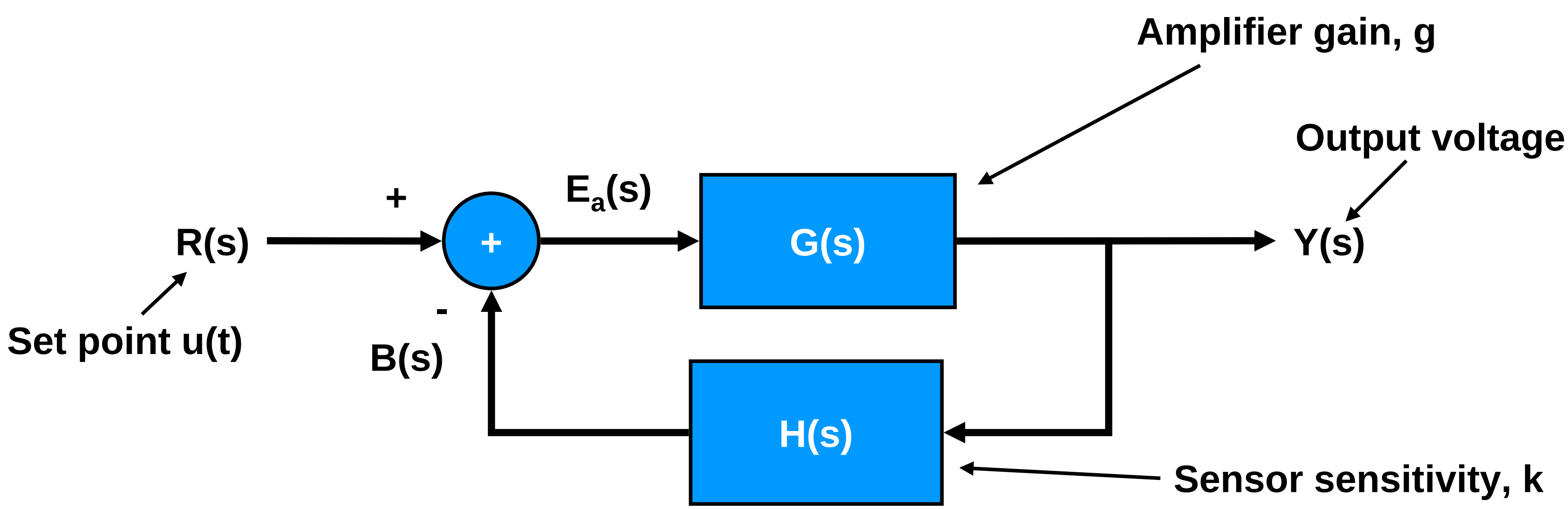

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Then:

$$\lim_{t\to\infty} e_a(t) = \lim_{s\to 0} \frac{1}{\left(1 + g\frac{k}{s}\right)} = \frac{s}{s + gk} = 0$$

Remember

$$\frac{1}{s} \equiv \int dt$$

Integration of error signal allows steady state error to go to zero

Next time

- Proportional, Integral and Derivative control systems