Design IV

E232 Spring 07

Class 23

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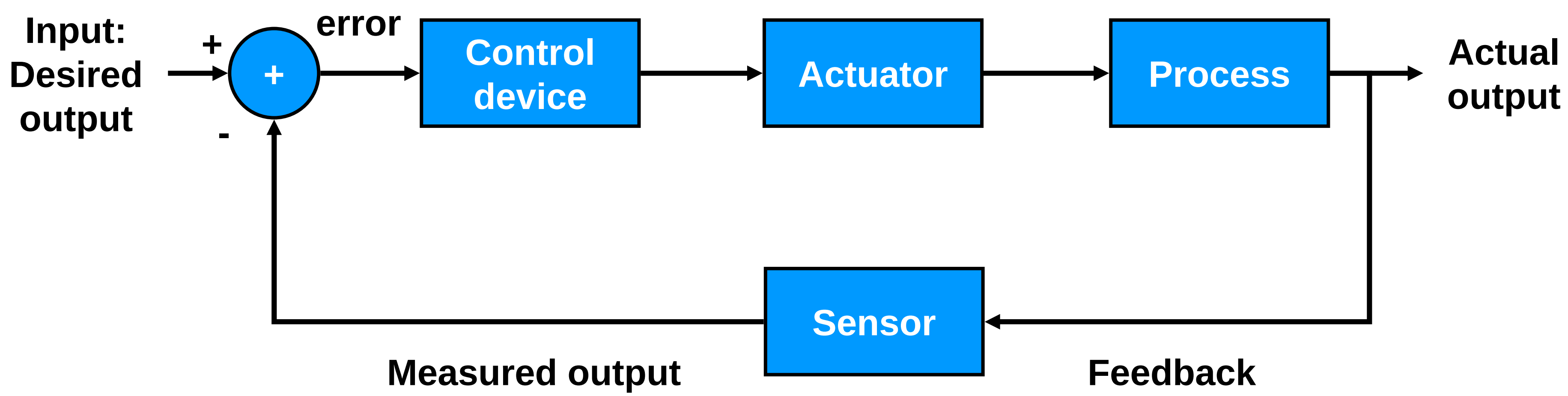
Negative Feedback Control System



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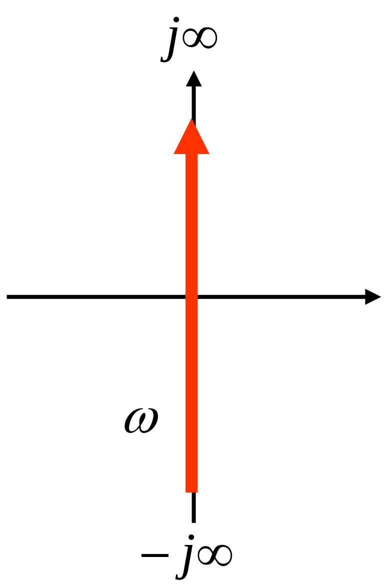
Generalizing The Fourier Transform

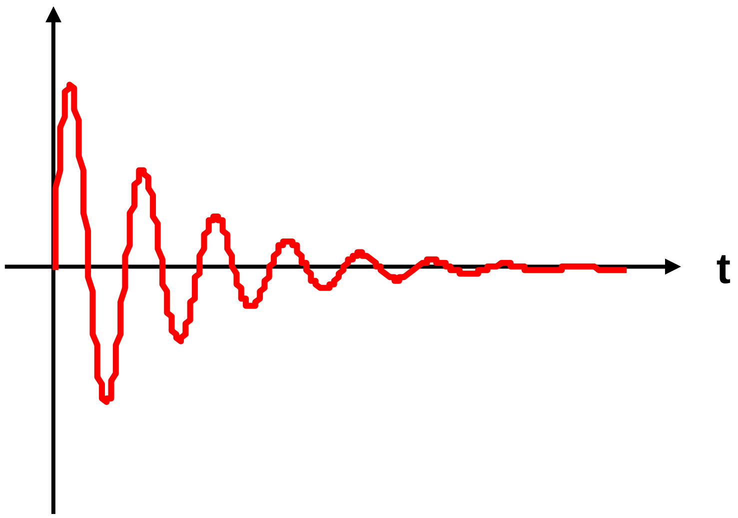
- The Fourier Transform works well with sinusoidal and oscillatory signals
- The Fourier Integral inherently assumes the signal lies somewhere on the jω axis

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$



- But signals in control systems generally exhibit damped or decaying behavior, which the Fourier Transform cannot readily represent



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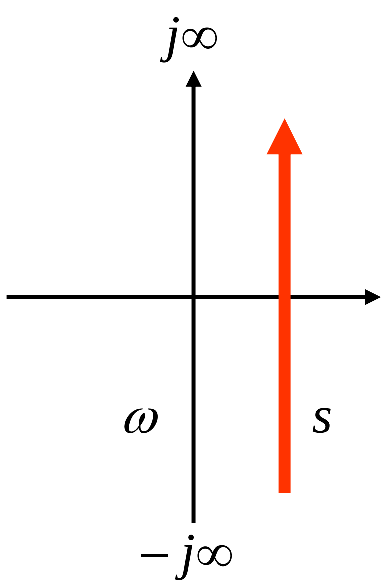
Generalizing The Fourier Transform: The Laplace Transform

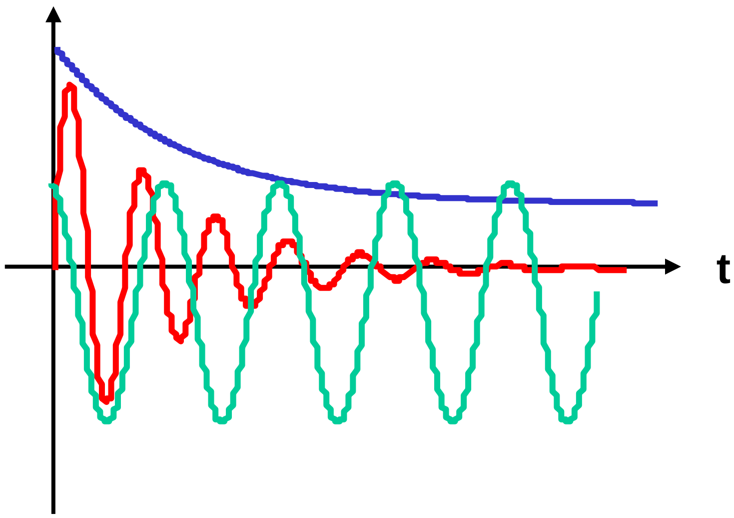
- The Laplace Transform is a generalization of the Fourier Transform with a transform operator that represents oscillatory as well as decaying oscillations

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$$



- The Laplace Transform can deal with a wider variety of signals than the Fourier Transform can.



- The Laplace Transform provides a straightforward way to transform differential equations into algebraic equations, which can be more easily solved.

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Today's topics

- Control systems
  - Mathematical analysis

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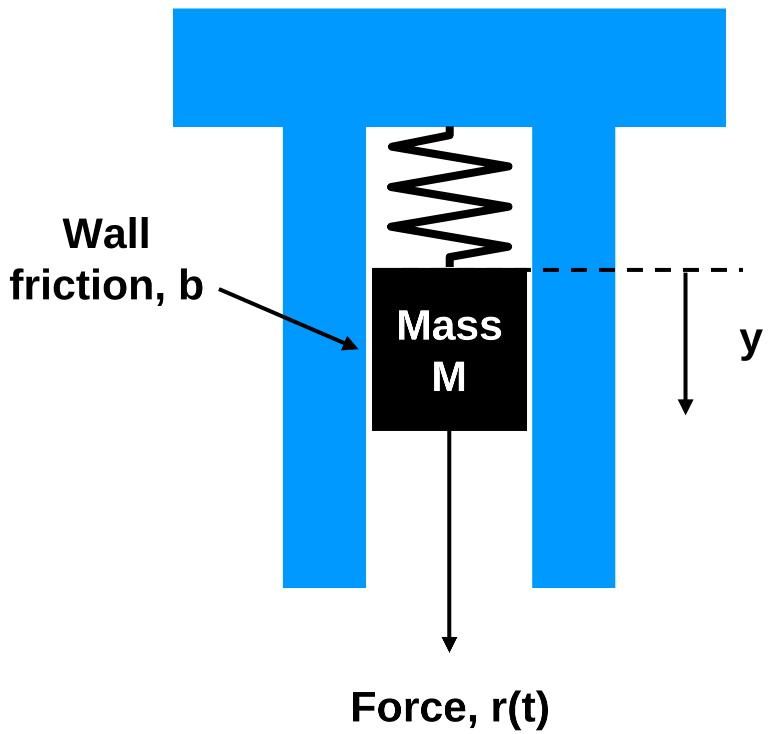
Analyzing Dynamic Systems – Differential Equations



$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

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Laplace Transform

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds = \mathscr{L}^{-1}\{F(s)\}$$

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt = \mathscr{L}\{f(t)\}$$

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Sample Laplace Transform Pairs

f(t)	F(s)
Unit impulse, $\delta(t)$	1
Unit step, u(t)	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{(s+a)}{(s+a)^2+\omega^2}$

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Equivalence of Laplace Operators

$$s \equiv \frac{d}{dt}$$

$$\frac{1}{s} \equiv \int_{0-}^{t} dt$$

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Using the Laplace Transform to Replace Differential Equations

- Original differential equation

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

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Using the Laplace Transform to Replace Differential Equations

- Substituting for derivatives:

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

$$M\left(s^2Y(s)-sy(0-)-\frac{dy(0-)}{dt}\right)+b\left(sY(s)-y(0-)\right)+kY(s)=R(s)$$

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Using the Laplace Transform to Replace Differential Equations

- Assume initial conditions:

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

$$M\left(s^2Y(s)-sy(0-)-\frac{dy(0-)}{dt}\right)+b\left(sY(s)-y(0-)\right)+kY(s)=R(s)$$

$$r(t)=0 \qquad y(0-)=y_0 \qquad \left.\frac{dy}{dt}\right|_{t=0-}=0$$

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Using the Laplace Transform to Replace Differential Equations

- Simplify:

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

$$M\left(s^2Y(s)-sy(0-)-\frac{dy(0-)}{dt}\right)+b\left(sY(s)-y(0-)\right)+kY(s)=R(s)$$

$$r(t)=0 \qquad y(0-)=y_0 \qquad \left.\frac{dy}{dt}\right|_{t=0-}=0$$

$$Ms^2Y(s)-Msy_0+bsY(s)-by_0+kY(s)=0$$

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Using the Laplace Transform to Solve Differential Equations

- Rearrange to solve for Y(s)

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

$$Y(s) = \frac{(Ms+b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}$$

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Using the Laplace Transform to Solve Differential Equations

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Determines
"characteristic equation"

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Determines
“characteristic equation”

- Example, let k/M=5, b/M=6

$$Y(s) = \frac{(s+\frac{b}{M})y_0}{s^2+\frac{b}{M}s+\frac{k}{M}} = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{p(s)}{q(s)}$$

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Using the Laplace Transform to Solve Differential Equations

- "Poles" and "zeroes" of Y(s)

At s=-6, Y(s)=0

$$Y(s) = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{p(s)}{q(s)}$$

At s=-1 or s=-5, Y(s) increases without bound

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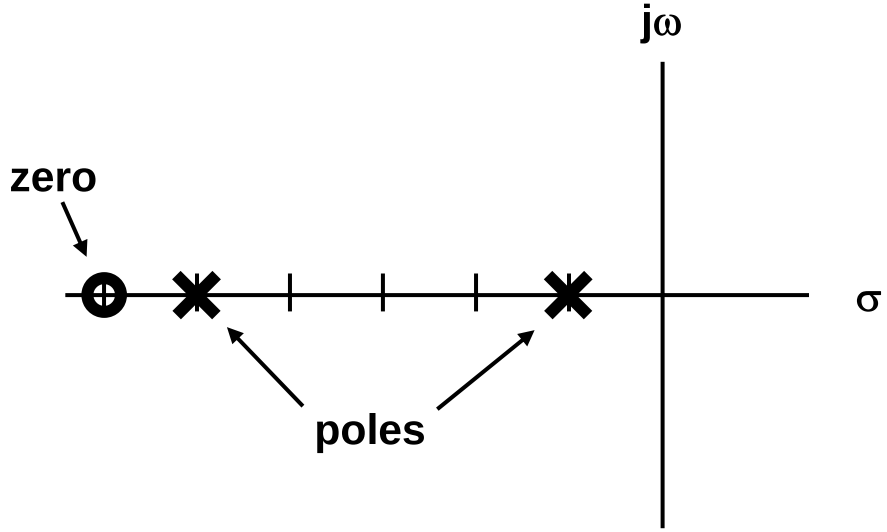

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Using the Laplace Transform to Solve Differential Equations

- Partial fraction expansion of Y(s), assume $y_0=1$

$$Y(s)=\frac{(s+6)y_0}{(s+5)(s+1)}=\frac{k_1}{s+1}+\frac{k_2}{s+5}$$

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Using the Laplace Transform to Solve Differential Equations

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$$Y(s)=\frac{(s+6)y_0}{(s+5)(s+1)}=\frac{k_1}{s+1}+\frac{k_2}{s+5}$$

$$k_1=\left.\frac{(s-s_1)\,p(s)}{q(s)}\right|_{s=s_1}=\left.\frac{(s+1)(s+6)}{(s+5)(s+1)}\right|_{s=-1}=\frac{5}{4}$$

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Using the Laplace Transform to Solve Differential Equations

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$$k_2=\left.\frac{(s-s_2)p(s)}{q(s)}\right|_{s=s_2}=\left.\frac{(s+5)(s+6)}{(s+5)(s+1)}\right|_{s=-5}=\frac{1}{-4}=-\frac{1}{4}$$

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Using the Laplace Transform to Solve Differential Equations

- Partial fraction expansion of Y(s), assume $y_0=1$

$$Y(s)=\frac{(s+6)}{(s+5)(s+1)}=\frac{5/4}{s+1}+\frac{-1/4}{s+5}$$

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Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$Y(s) = \frac{(s+6)}{(s+5)(s+1)} = \frac{5/4}{s+1} + \frac{-1/4}{s+5}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1} + \frac{-1/4}{s+5}\right\}$$

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Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

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$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1/4}{s+5}\right\}$$

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Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

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Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1/4}{s+5}\right\}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

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Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1/4}{s+5}\right\}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$$

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Using the Laplace Transform to Solve Differential Equations

- Final value theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

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Using the Laplace Transform to Solve Differential Equations

- Final value theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{s(s+6)}{(s+5)(s+1)} = \frac{0 \cdot 6}{5 \cdot 1} = 0$$

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Behavior of Under-damped System

$$Y(s)=\frac{\left(s+\frac{b}{M}\right)y_0}{\left(s^2+\frac{b}{M}s+\frac{k}{M}\right)}=\frac{\left(s+2\zeta\omega_n\right)y_0}{\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)}$$

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Behavior of Under-damped System

$$Y(s) = \frac{\left(s + \frac{b}{M}\right) y_0}{\left(s^2 + \frac{b}{M}s + \frac{k}{M}\right)} = \frac{\left(s + 2\zeta\omega_n\right) y_0}{\left(s^2 + 2\zeta\omega_n s + \omega_n{}^2\right)}$$

Damping ratio **Natural frequency**

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Behavior of Under-damped System

$$Y(s)=\frac{\left(s+\frac{b}{M}\right)y_0}{\left(s^2+\frac{b}{M}s+\frac{k}{M}\right)}=\frac{\left(s+2\zeta\omega_n\right)y_0}{\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)}$$

Damping ratio **Natural frequency**

$$s_1,s_2=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

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Behavior of Under-damped System

$$Y(s)=\frac{\left(s+\frac{b}{M}\right)y_0}{\left(s^2+\frac{b}{M}s+\frac{k}{M}\right)}=\frac{\left(s+2\zeta\omega_n\right)y_0}{\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)}$$

Damping ratio **Natural frequency**

$$s_1, s_2=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

If ζ < 1

$$s_1, s_2=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$$

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Behavior of Under-damped System

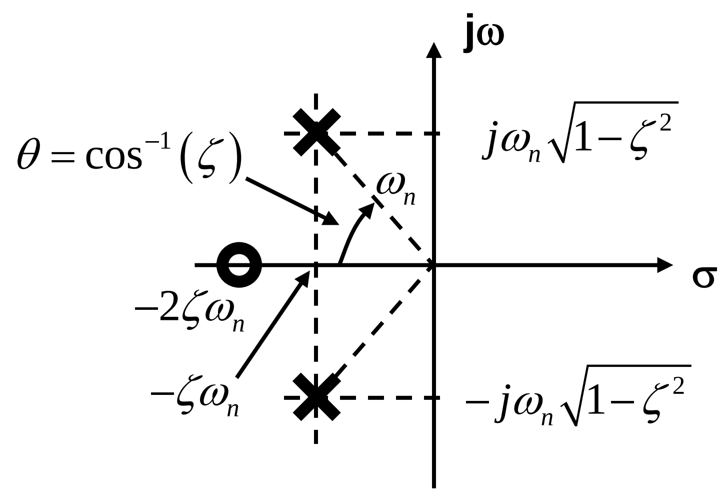
$$Y(s)=\frac{\left(s+\frac{b}{M}\right)y_0}{\left(s^2+\frac{b}{M}s+\frac{k}{M}\right)}=\frac{\left(s+2\zeta\omega_n\right)y_0}{\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)}$$

Damping ratio **Natural frequency**

$$s_1,s_2=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

If ζ < 1

$$s_1,s_2=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$$



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Next time

- Transfer functions of feedback control systems
- Steady state error of proportional control systems

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Homework 9

(1) (Dorf&Bishop problem E2.4)
(2) (Dorf&Bishop problem P2.8)

Problems can be found on WebCT under Course Content and Related Materials > Homework Problems

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