

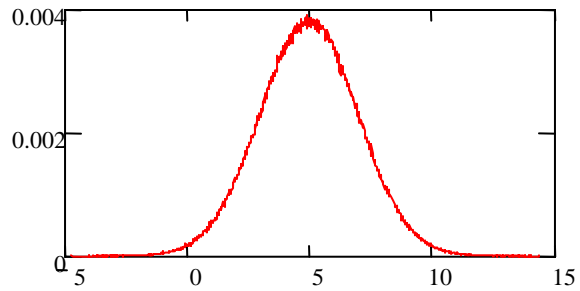
# Design IV

## E232 Spring 07

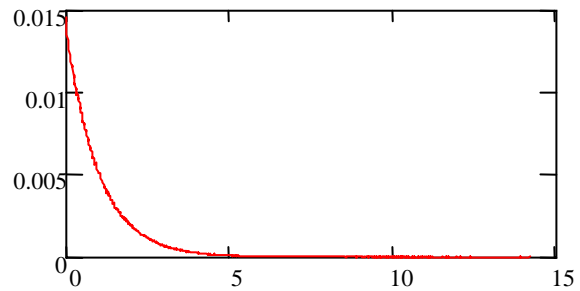
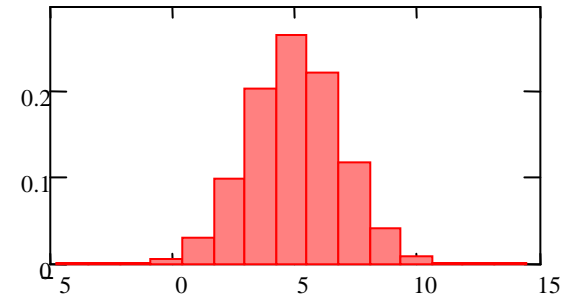
Class 12

Bruce McNair  
bmcnair@stevens.edu

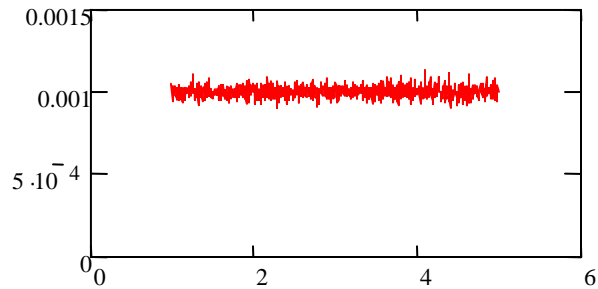
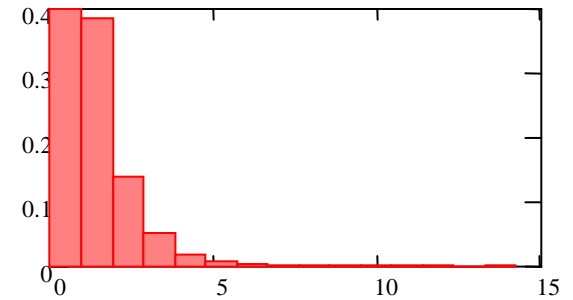
# Sample Distributions



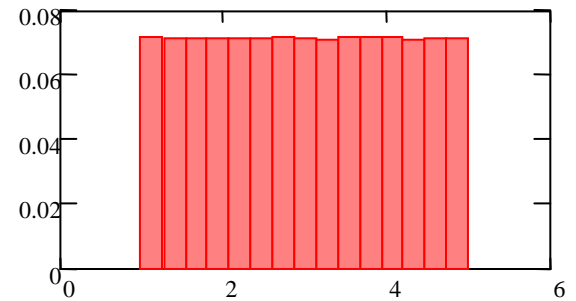
**Normal/  
Gaussian**



**Exponential**



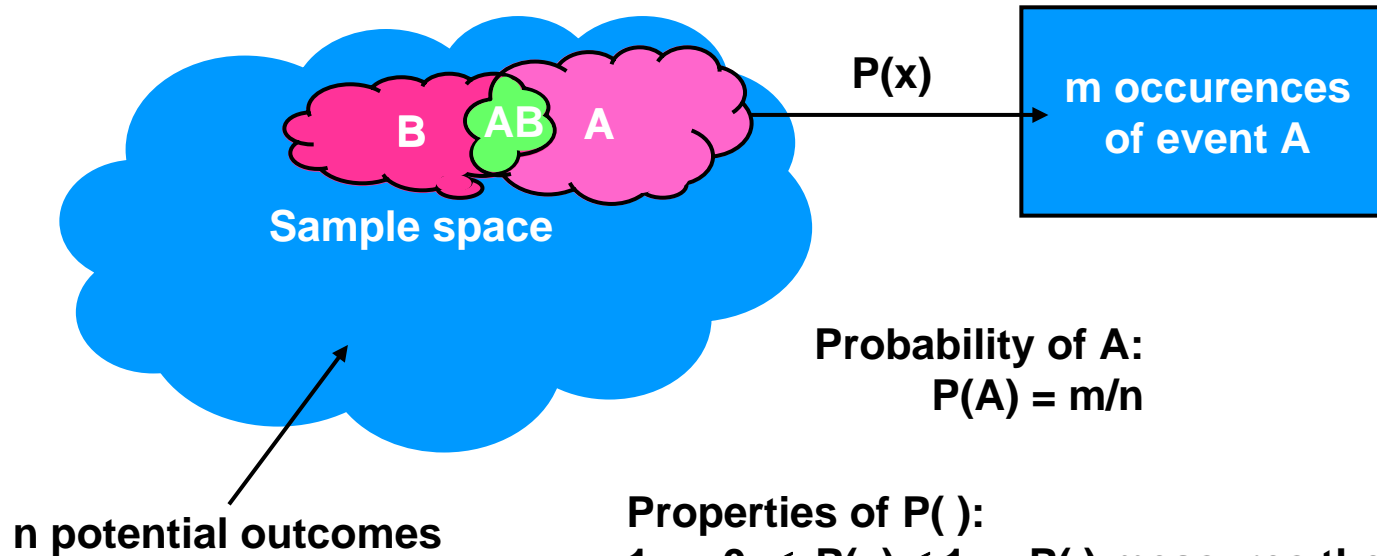
**Uniform**



**Continuous**

**Discrete**

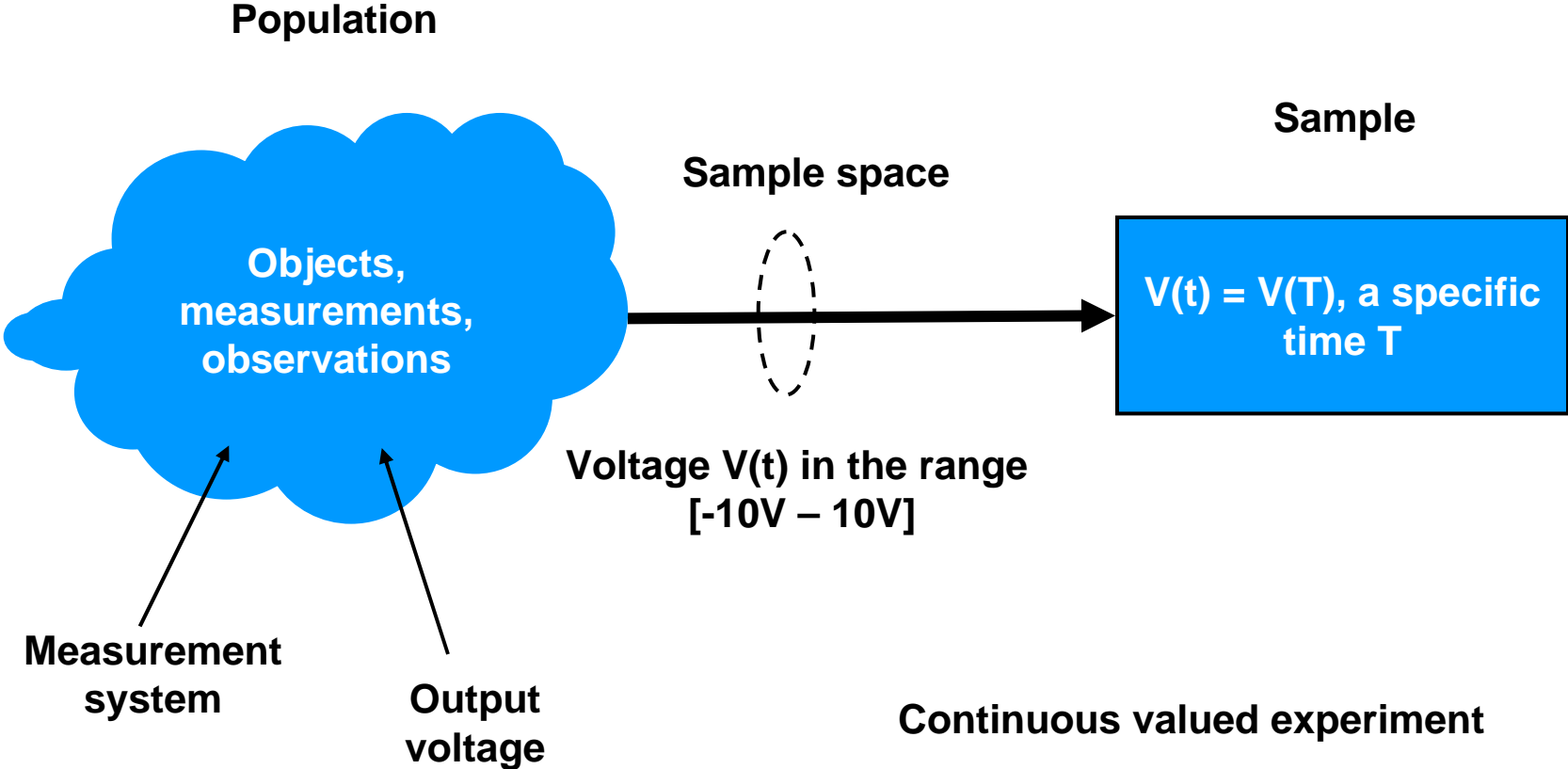
# Probability Measures



## Properties of $P(\ )$ :

1.  $0 \leq P(x) \leq 1$  -  $P(\ )$  measures the proportion of A
2. If A is certain,  $P(A) = 1$
3. If A is impossible,  $P(A) = 0$
4.  $\neg A$  is the complement of A.  $P(\neg A) = 1 - P(A)$
5. If A and B are mutually exclusive,  
 $P(A \text{ or } B) = P(A) + P(B)$
6. If A and B are independent events,  
 $P(AB) = P(A)P(B)$
7. Probability of either A or B (or both):  
 $P(\text{Union of A, B}) =$   
 $P(A \cup B) = P(A) + P(B) - P(AB)$

# Terminology

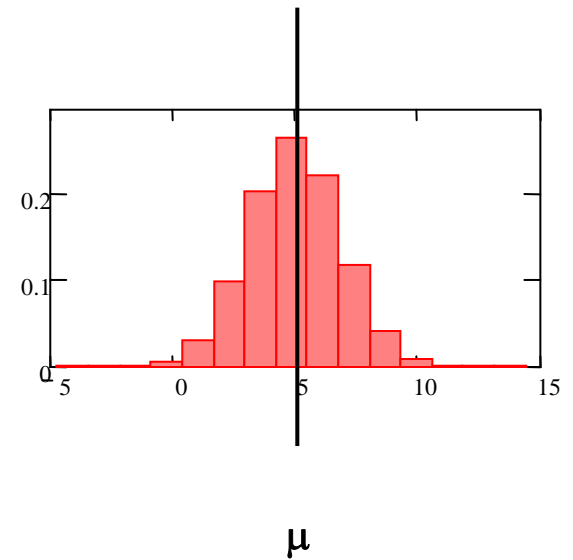


# Characterizing Distributions

- Mean, average, central tendency
- **Assume a set of n measurements**
- **Assume a population of N elements**
- **Assume measurement values,  $x_i$**

$$\mu = \frac{x_1 + x_2 + x_3 + \dots + x_N}{N} = \sum_{i=1}^N \frac{x_i}{N}$$

**Mean of a population**

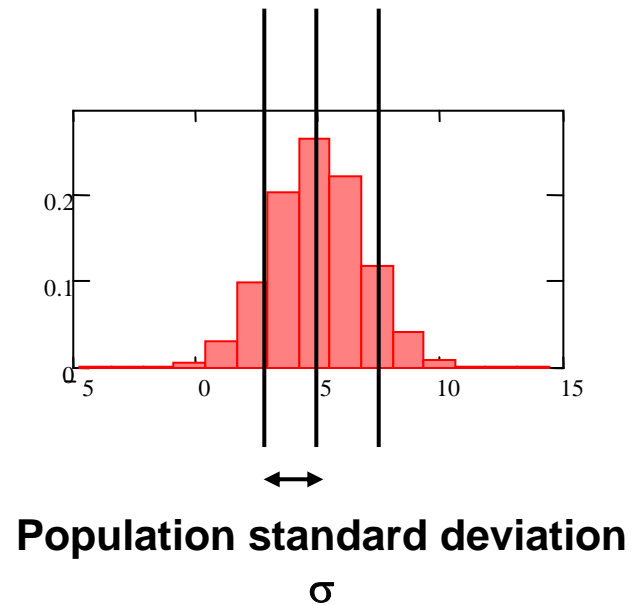


# Characterizing Distributions

- Measuring deviation
- Assume a set of n measurements
- Assume a population of N elements
- Assume measurement values,  $x_i$

$$\sigma = \sqrt{\sum_{i=1}^N \frac{(x_i - \mu)^2}{N}}$$

**Standard deviation from mean of population  
(note resemblance to RMS)**



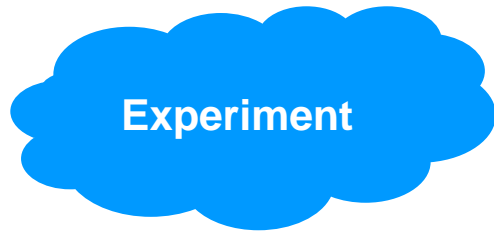
# Today's topics

- More Statistical Analysis of Experimental Data
  - Probability distribution functions

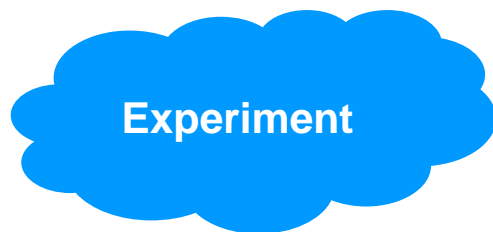


# Probability Distribution Functions

- Empirical distributions

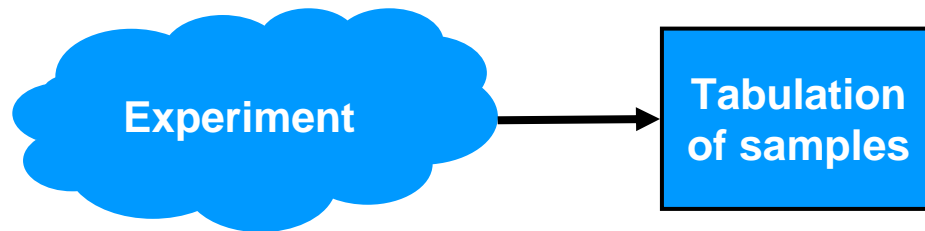


- Mathematical distributions

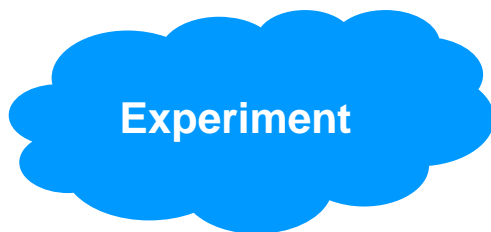


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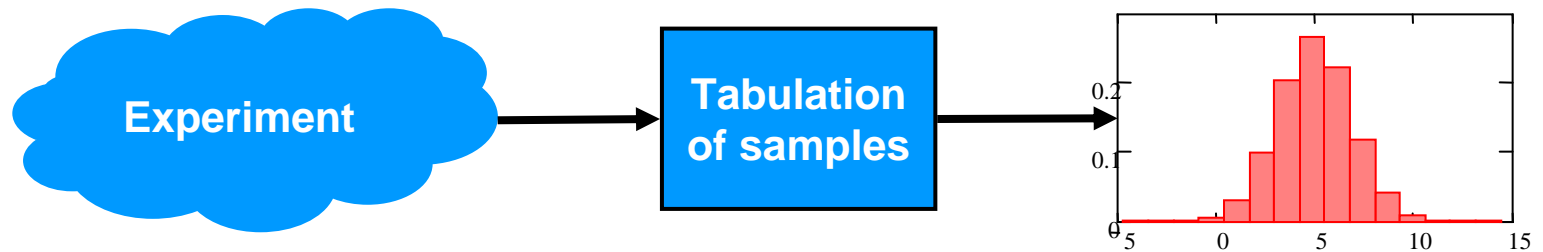


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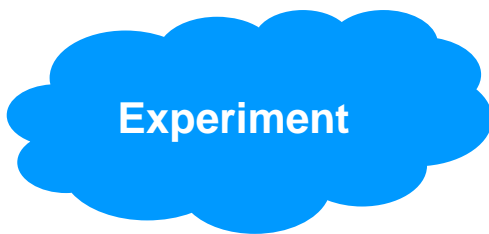


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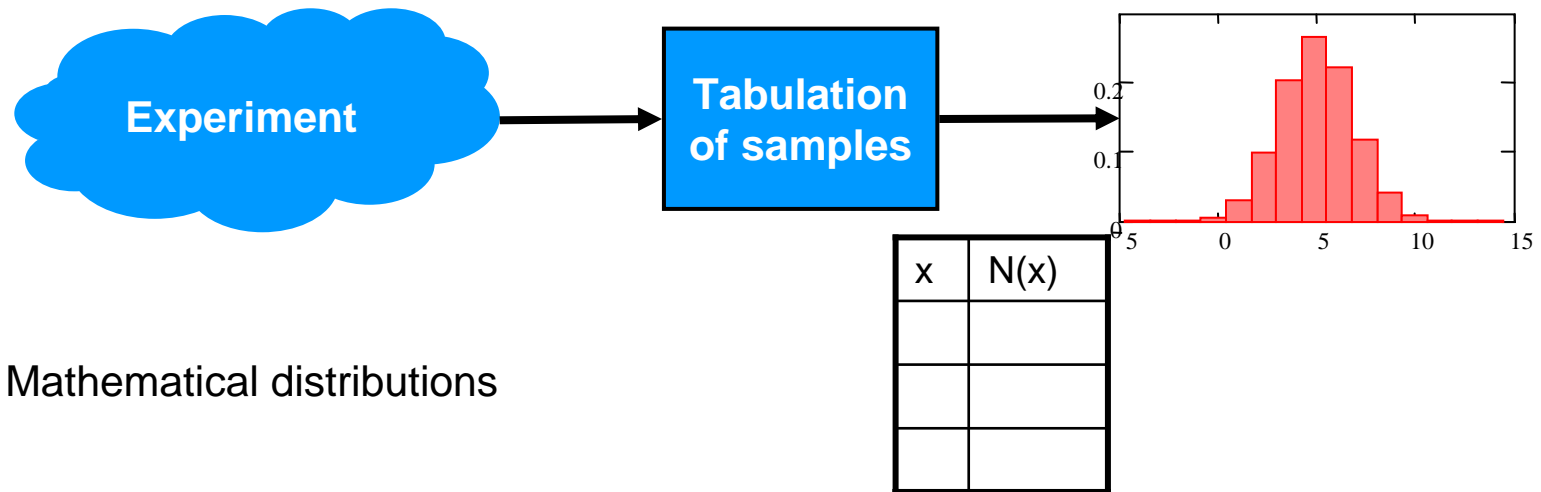


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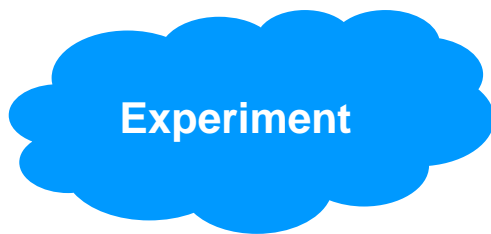


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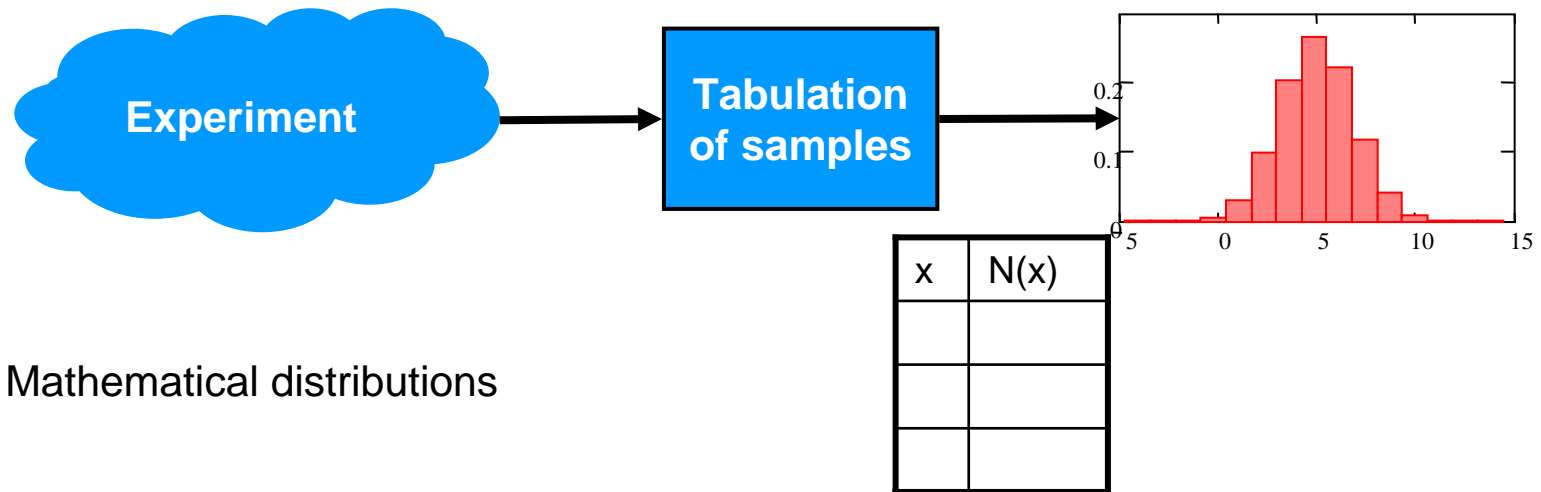


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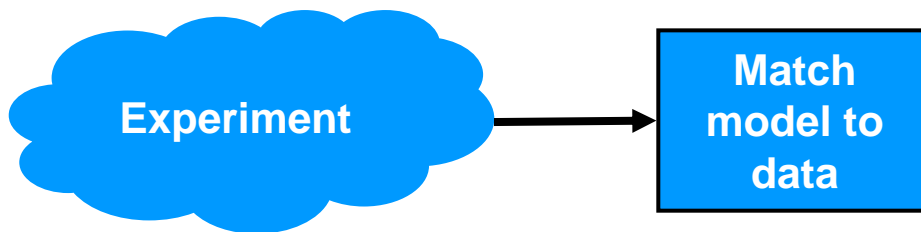


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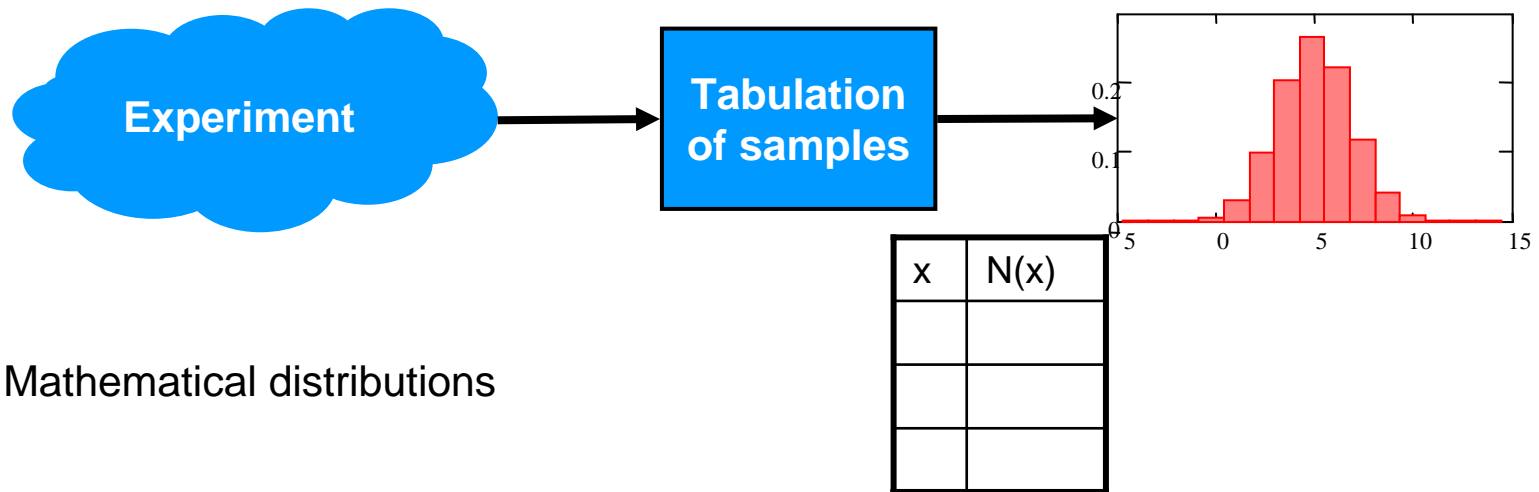


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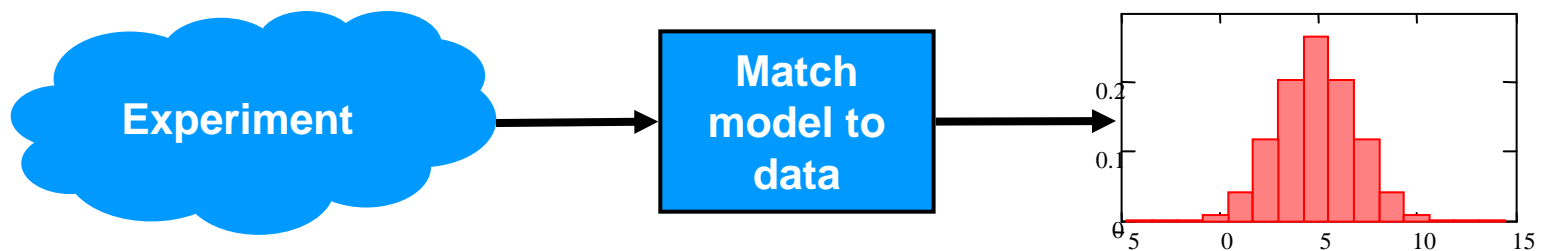


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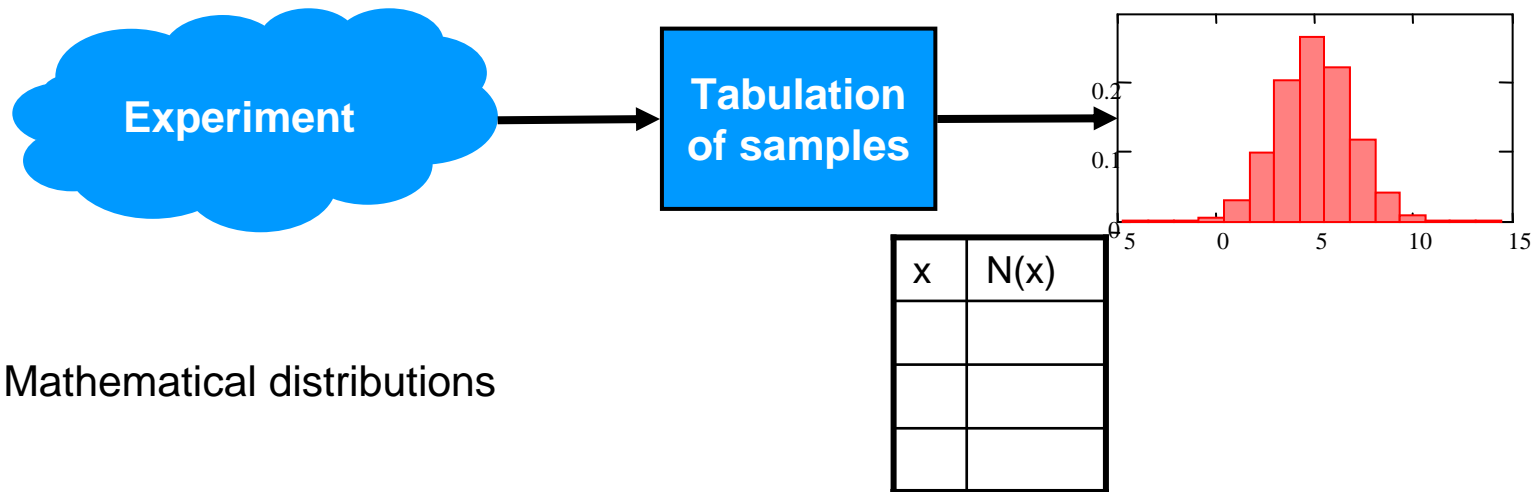


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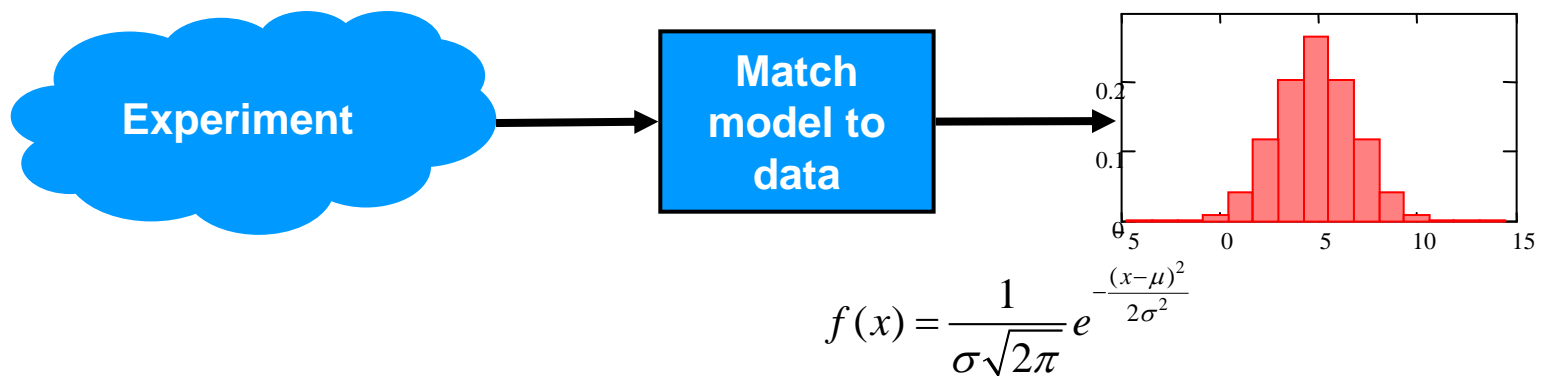


# Probability Distribution Functions

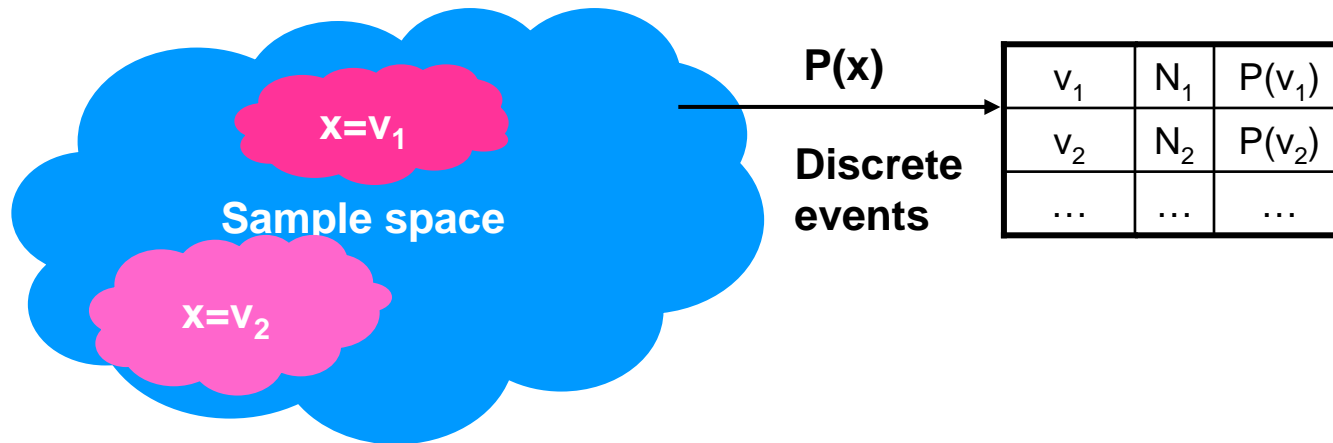
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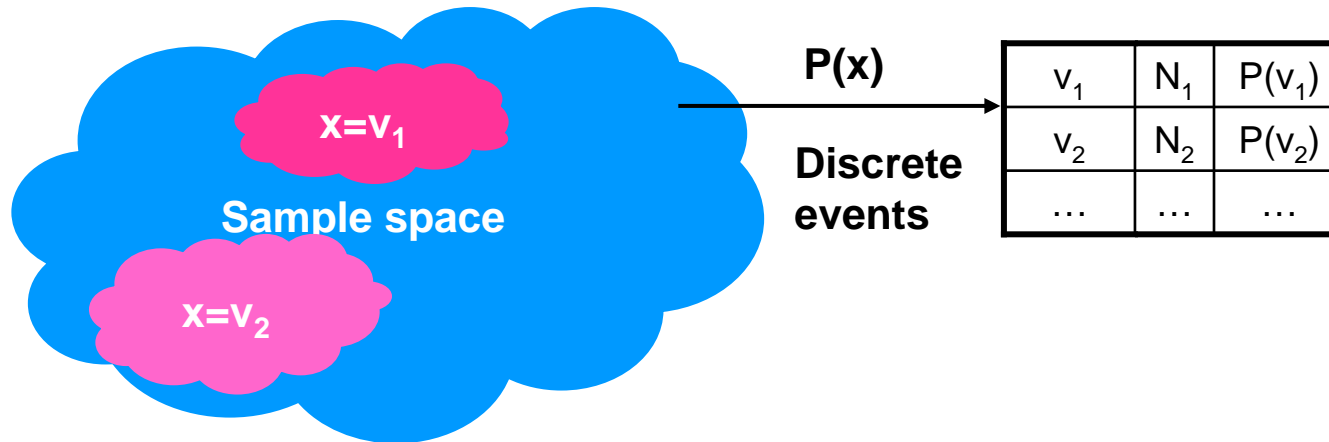
- Mathematical distributions



# Probability Mass Function



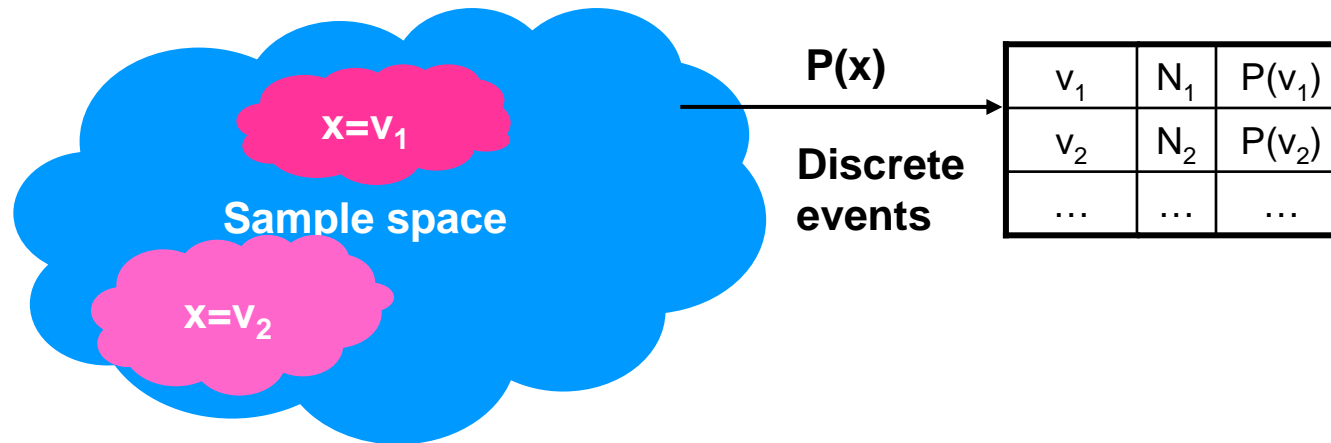
# Probability Mass Function



$$\sum_{i=1}^n P(x_i) = 1$$

- The sum of the probabilities of all possible events = 1

# Probability Mass Function



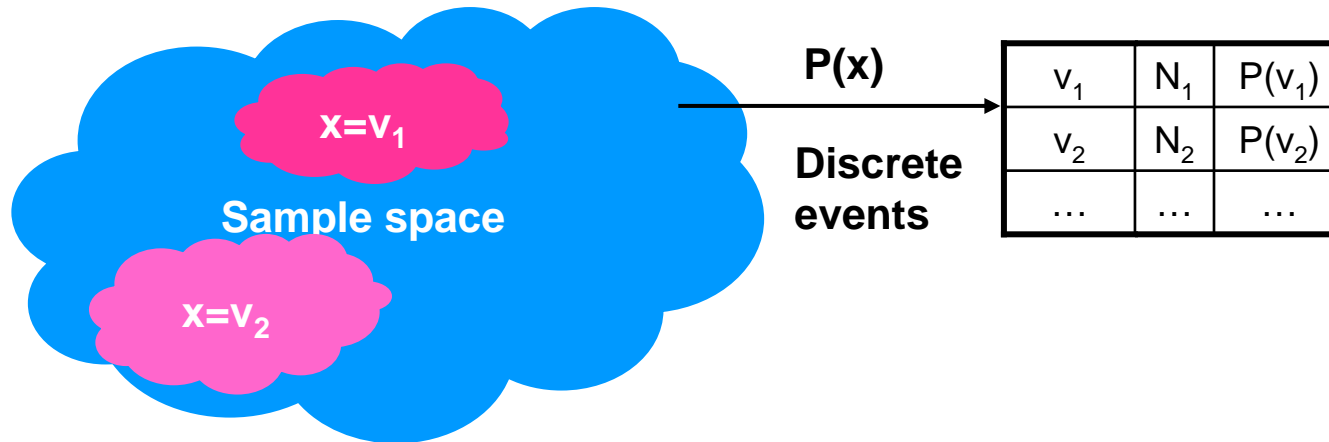
$$\sum_{i=1}^n P(x_i) = 1$$

• The sum of the probabilities of all possible events = 1

$$\mu = \sum_{i=1}^n x_i P(x_i)$$

• mean = weighted sum of values

# Probability Mass Function



$$\sum_{i=1}^n P(x_i) = 1$$

- The sum of the probabilities of all possible events = 1

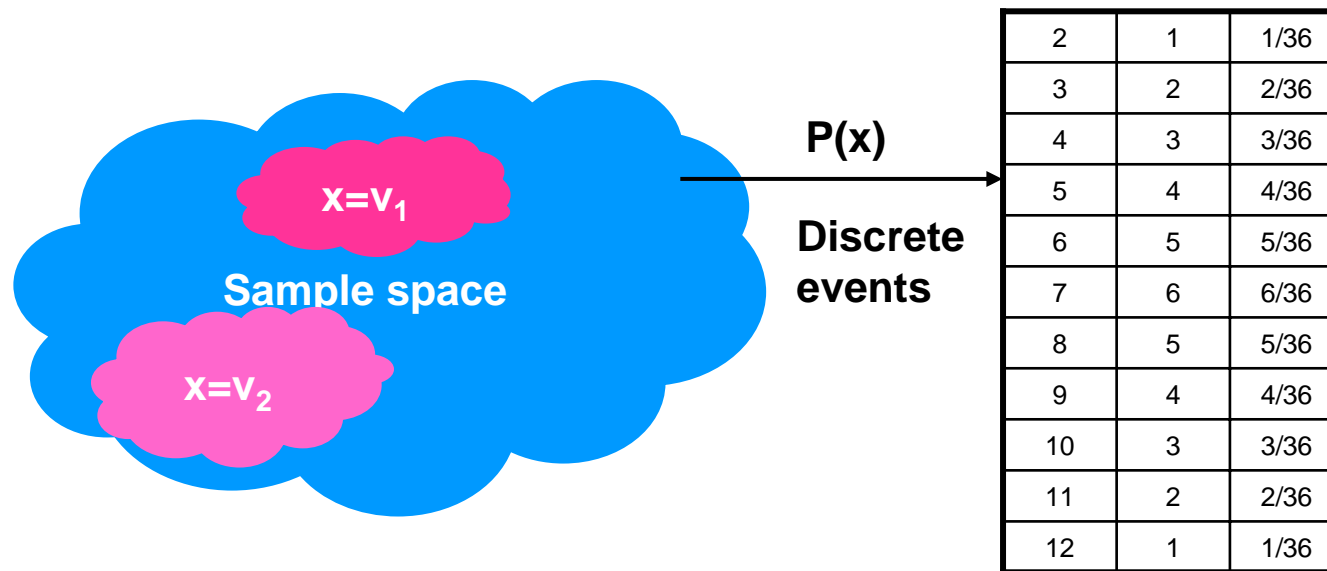
$$\mu = \sum_{i=1}^n x_i P(x_i)$$

- mean = weighted sum of values

$$\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 P(x_i)$$

- variance

# Probability Mass Function



**Experiment:**  
Number of dots  
showing on a pair  
of dice

$$\sum_{i=1}^n P(x_i) = \frac{1}{36} + \frac{2}{36} + \dots = \frac{36}{36} = 1$$

• The sum of the probabilities  
of all possible events = 1

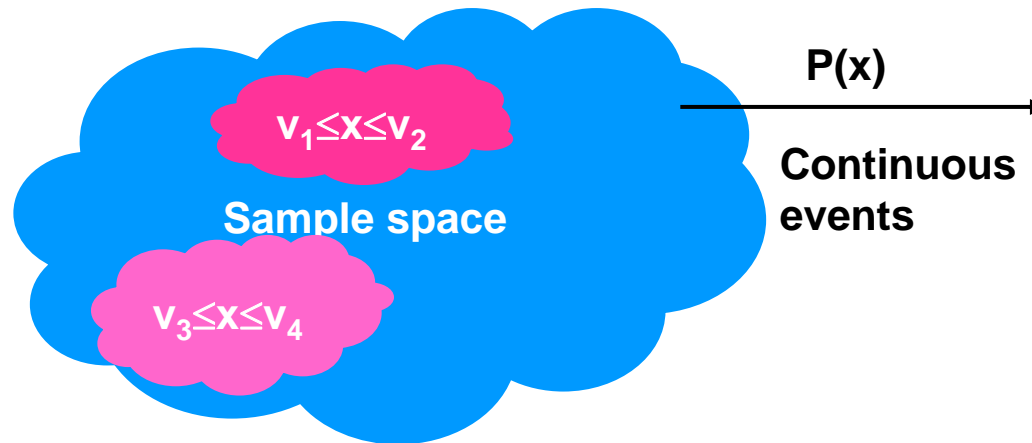
$$\mu = \sum_{i=1}^n x_i P(x_i) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \dots = 7$$

• mean = weighted sum of  
values

$$\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 P(x_i) = (-5)^2 \cdot \frac{1}{36} + (-4)^2 \cdot \frac{2}{36} + \dots = 5.83\bar{3}$$

• variance

# Probability Density Function

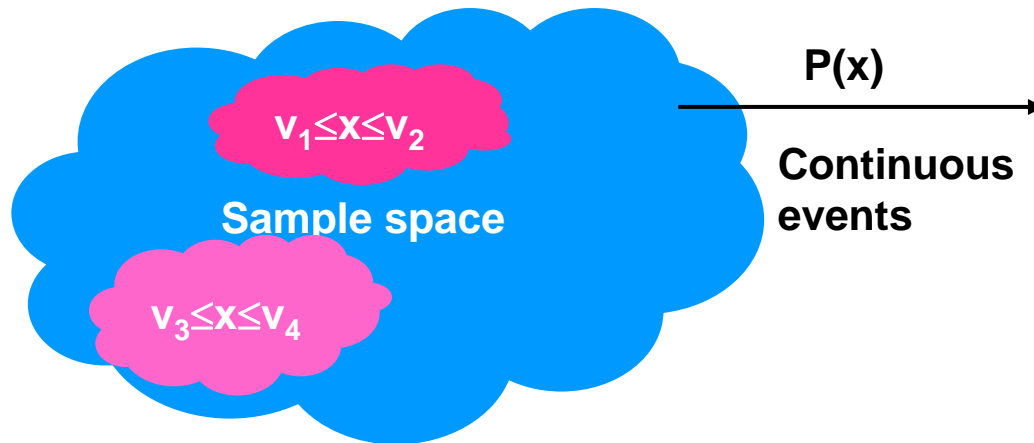


**Experiment:**  
Temperature  
measured by sensor

$$P(x = x_0) = P(x_0 \leq x \leq x_0) = 0$$

• **Discrete values have zero density**

# Probability Density Function



**Experiment:**  
Temperature  
measured by sensor

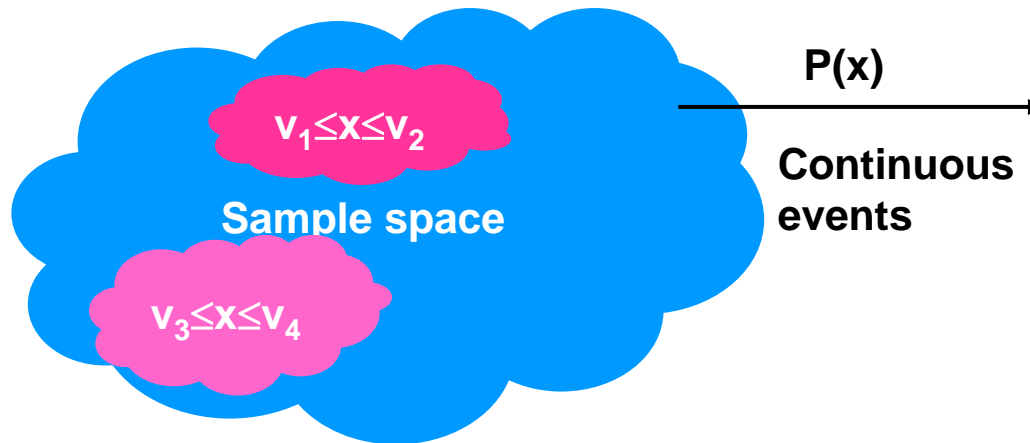
$$P(x = x_0) = P(x_0 \leq x \leq x_0) = 0$$

$$P(x_i \leq x \leq x_i + dx) = f(x_i)dx$$

- Discrete values have zero density

- $f(x_i)$  measures probability that  $x$  will be within  $dx$  of  $x_i$

# Probability Density Function



**Experiment:**  
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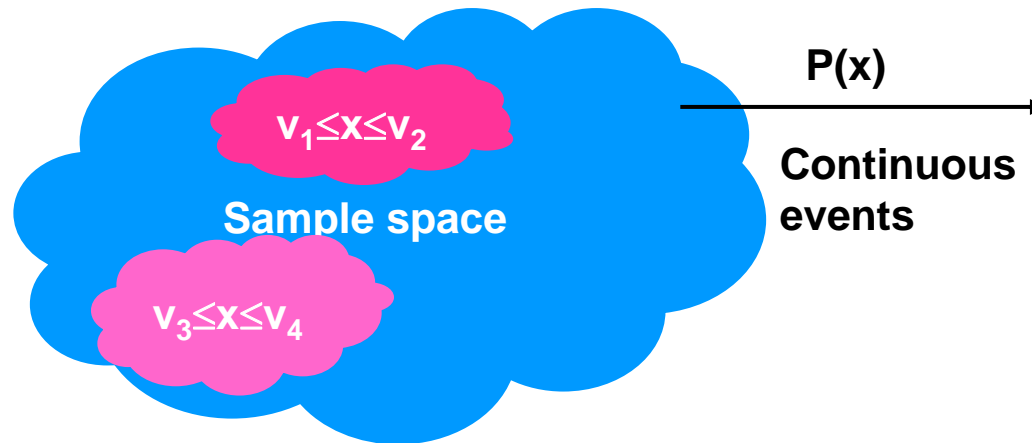
$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

- Discrete values have zero density

- $f(x_i)$  measures probability that  $x$  will be within  $dx$  of  $x_i$

- Probability that  $x$  will be in a given range

# Probability Density Function

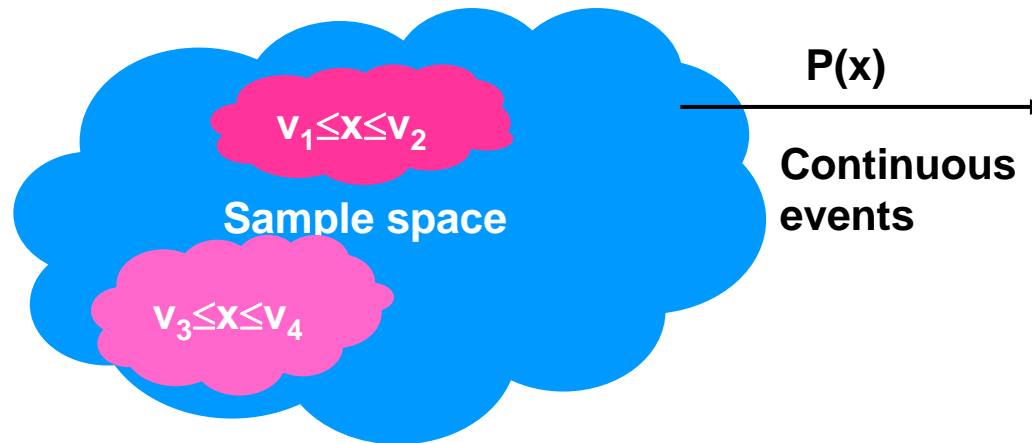


**Experiment:  
Temperature  
measured by sensor**

$$P(-\infty \leq x \leq \infty) = 1$$

- $x$  must have some value

# Probability Density Function



**Experiment:  
Temperature  
measured by sensor**

$$P(-\infty \leq x \leq \infty) = 1$$

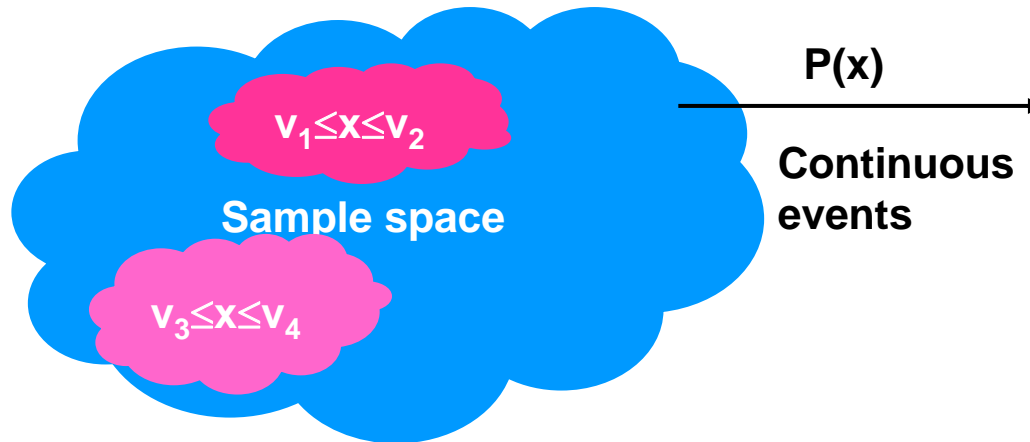
• **x must have some value**

$$\mu = \sum_{i=1}^n x_i P(x_i)$$

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

• **mean is defined similarly to  
discrete variables**

# Probability Density Function



**Experiment:  
Temperature  
measured by sensor**

$$P(-\infty \leq x \leq \infty) = 1$$

•  $x$  must have some value

$$\mu = \sum_{i=1}^n x_i P(x_i)$$

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

• mean is defined similarly to discrete variables

$$\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 P(x_i) \quad \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

• variance is defined similarly to discrete variables

# Probability Density Function - Example

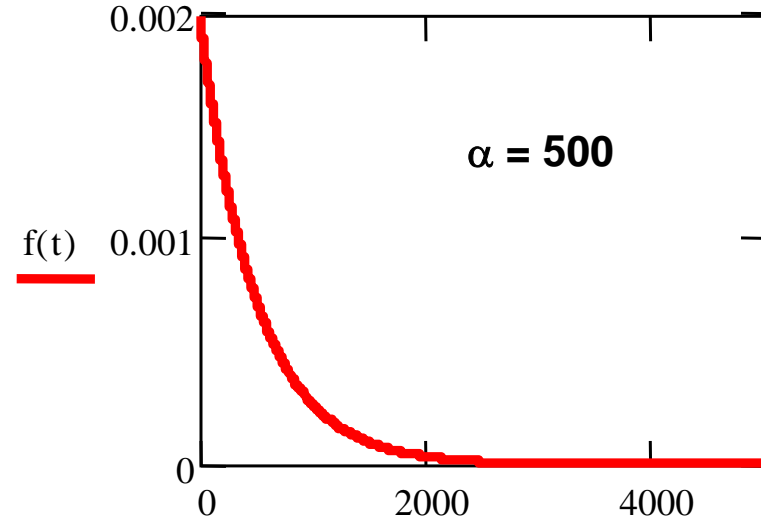
- Electrical component lifetime:

**Probability that component with average lifetime  $\alpha$  fails before time T:**

$$P(t \leq T) = \int_0^T f(t) dt$$

$$f(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\alpha} e^{-\frac{t}{\alpha}} & t \geq 0 \end{cases}$$

$$P(t \leq T) = \int_0^T \frac{1}{\alpha} e^{-\frac{t}{\alpha}} dt$$



# Probability Density Function - Example

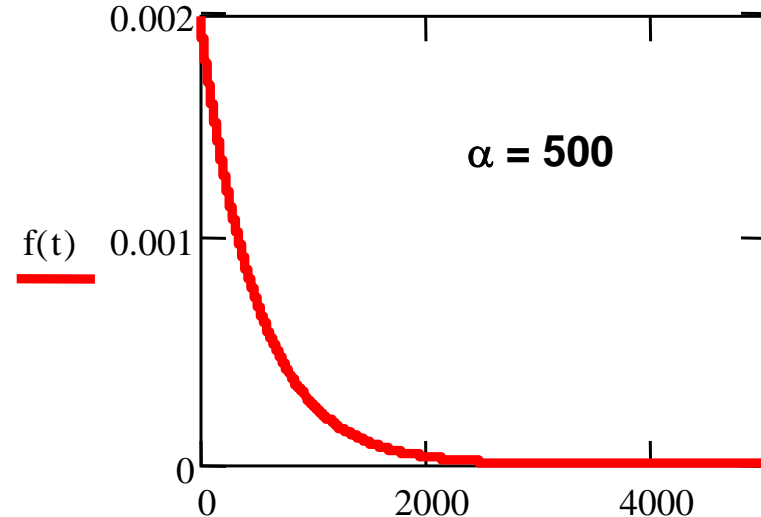
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**What is the probability that a light bulb with a 500 hour expected lifetime will fail within the first 100 hours?**

# Probability Density Function - Example

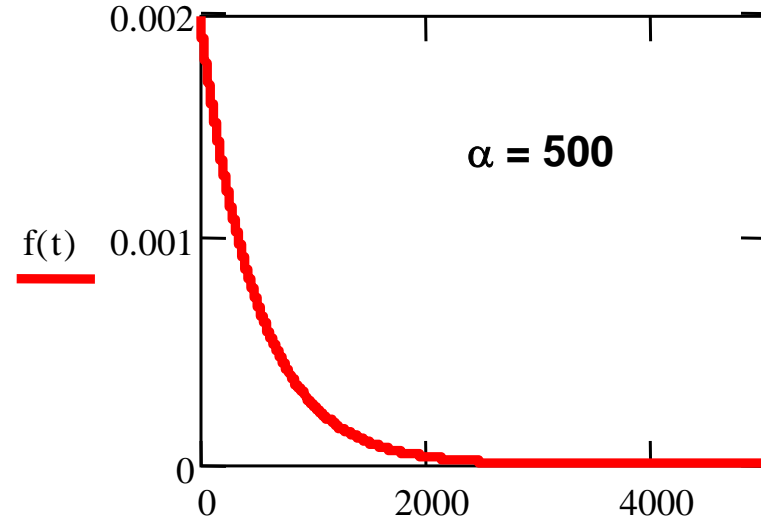
- Electrical component lifetime – exponential distribution:

**Probability that component with average lifetime  $\alpha$  fails before time T:**

$$P(t \leq T) = \int_0^T f(t) dt$$

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$$P(t \leq T) = \int_0^T \frac{1}{\alpha} e^{-\frac{t}{\alpha}} dt$$



**What is the probability that a light bulb with a 500 hour expected lifetime will fail within the first 100 hours?**

$$\int_0^{100} \frac{1}{500} \cdot e^{-\frac{t}{500}} dt = 0.221$$

# Cumulative Distribution Function

**Continuous random variable:**

$$P(rv \leq x) = \int_{-\infty}^x f(t)dt \doteq F(x)$$

**Discrete random variable:**

$$P(rv \leq x_i) = \sum_{j=1}^i P(x_j)$$

# Cumulative Distribution Function

**Continuous random variable:**

$$P(rv \leq x) = \int_{-\infty}^x f(t)dt \doteq F(x)$$

**Discrete random variable:**

$$P(rv \leq x_i) = \sum_{j=1}^i P(x_j)$$

## **Properties of C.D.F**

$$P(a < x \leq b) = F(b) - F(a)$$

$$P(x > a) = 1 - F(a)$$

# Useful P.D.F.'s – Binomial Distribution

- A set of discrete random variables can have two possible outcomes: “success” or “failure”
  - A number of trials are performed, each “succeeds” or “fails”
  - $P(\text{success})$  remains constant during trials
  - $n$  independent trials
- What is the probability of having exactly  $r$  successes?

# Useful P.D.F.'s – Binomial Distribution

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  - A number of trials are performed, each “succeeds” or “fails”
  - P(success) remains constant during trials
  - n independent trials
- What is the probability of having exactly r successes?

$$P(r) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$\binom{n}{r} \doteq \frac{n!}{r!(n-r)!}$$

## Useful P.D.F.'s – Binomial Distribution

- A machining process creates a part with a dimension that has a random component. 85% of the parts lie within the required range of dimensions, but 15% are out of spec. What is the probability that 80 of 100 randomly chosen components will be acceptable?

## Useful P.D.F.'s – Binomial Distribution

- A machining process creates a part with a dimension that has a random component. 85% of the parts lie within the required range of dimensions, but 15% are out of spec. What is the probability that 80 of 100 randomly chosen components will be acceptable?

$$P(r) = \binom{n}{r} p^r (1-p)^{n-r} = \left( \frac{n!}{r!(n-r)!} \right) p^r (1-p)^{n-r}$$

$$P(r) = \left( \frac{100!}{80!(100-80)!} \right) (0.85)^{80} (0.15)^{100-80} = .04$$

# Next time

- More On Statistical Analysis of Experimental Data
  - Other useful PDF's
    - Gaussian
    - Poisson
  - Correlation