Design IV

E232 Spring 07

Class 10

Bruce McNair
bmcnair@stevens.edu

Measuring Similarity and Distance

- The projection of A onto B, where A&B are periodic functions with period T:

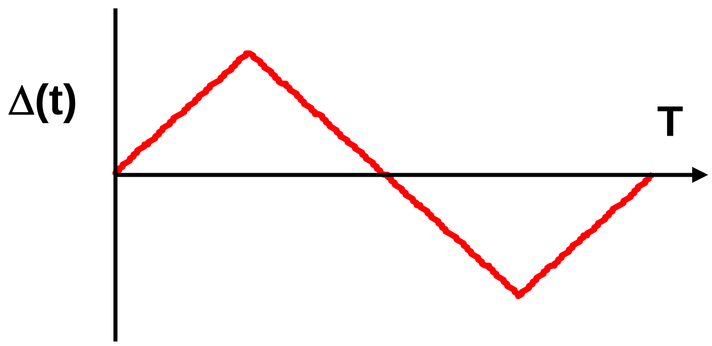
$$\langle A, B \rangle = \frac{1}{T} \int_{t}^{t+T} A(x) B(x) dx$$

- The length of A is the similarity of A with itself:

$$\langle A, A \rangle = | A |^2 = \frac{1}{T} \int_{t}^{t+T} A(x)^2 dx$$

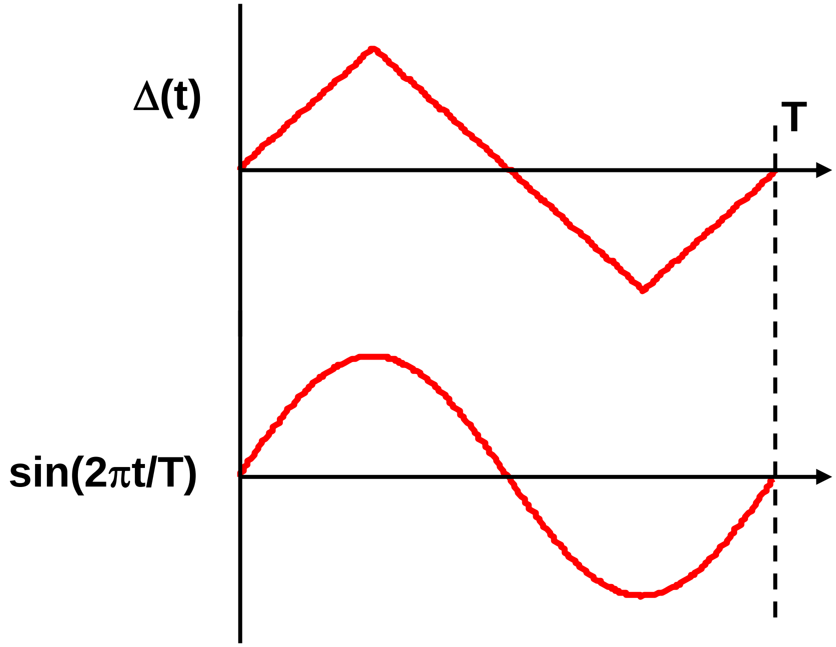
Spectral Analysis of Arbitrary Signals

- Consider a triangular wave with a period T:



Spectral Analysis of Arbitrary Signals

- Consider a triangular wave with a period T:

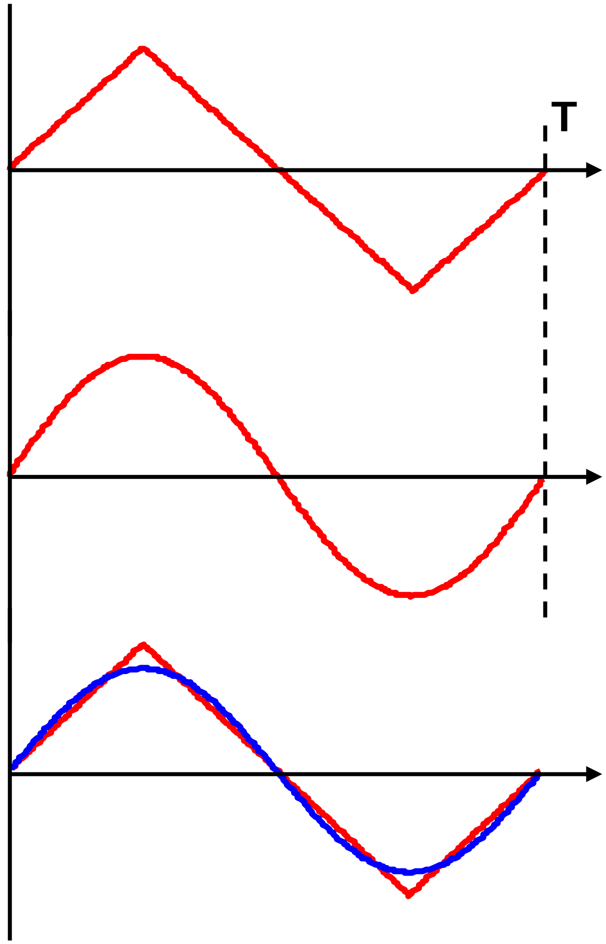


How similar is this signal to a sinusoid with the same period?

$$\frac{1}{\pi}\int_0^T \Delta(t)\sin\left(\frac{2\pi t}{T}\right)dt = 0.811$$

Spectral Analysis of Arbitrary Signals

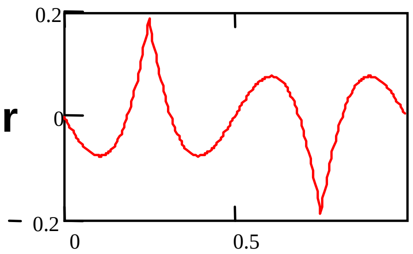
- Consider a triangular wave with a period T:



How similar is this signal to a sinusoid with the same period?

$$\frac{1}{\pi}\int_0^T \Delta(t)\sin\left(\frac{2\pi t}{T}\right)dt = 0.811$$

Error



Spectral Analysis With Arbitrary Signals

- Any well-behaved periodic signal f(t) can be represented as

$$f(t) = a_0 + \left(\sum_{i=1}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t \right) \right) + \left(\sum_{i=1}^{\infty} b_i \sin\left(i \frac{2\pi}{T} t \right) \right)$$

where

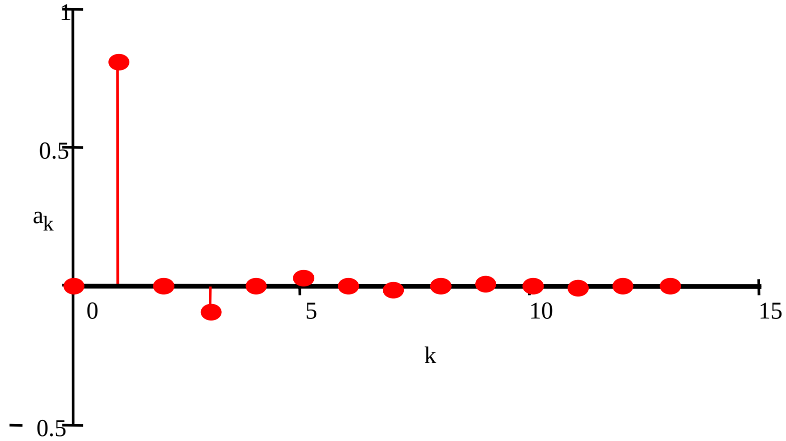
$$a_0 = \frac{1}{T} \int_0^T f(t)\,dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(i \frac{2\pi}{T} t \right) dt$$

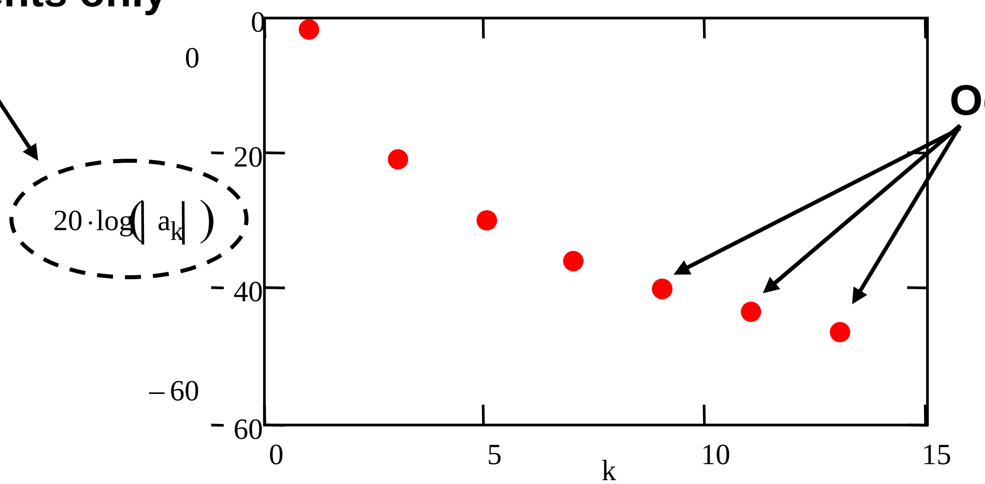
$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(i \frac{2\pi}{T} t \right) dt$$

Spectral Analysis With Arbitrary Signals

- Spectrum of triangular wave



Sine components only



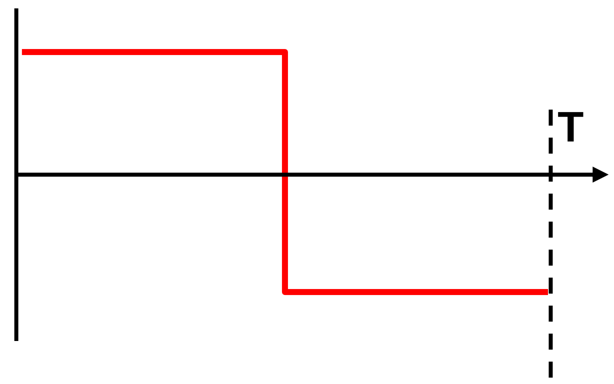
Odd harmonics only

Today's topics

- Computerized Data Acquisition
  - More on Fourier Transform and frequency domain analysis

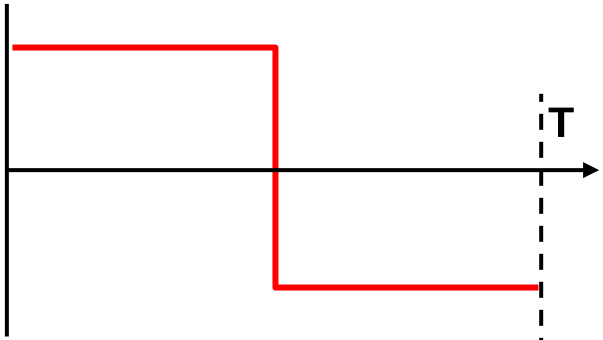
Spectrum Of A Few Common Signals

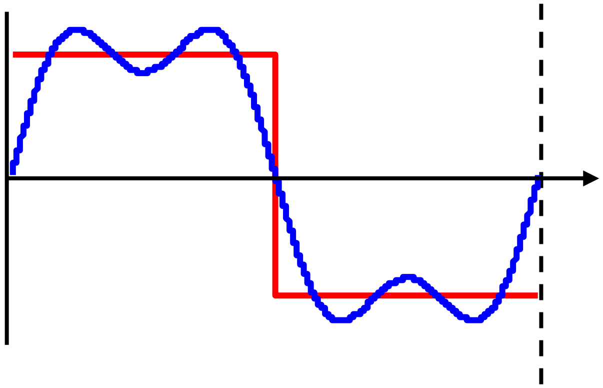
- Square wave



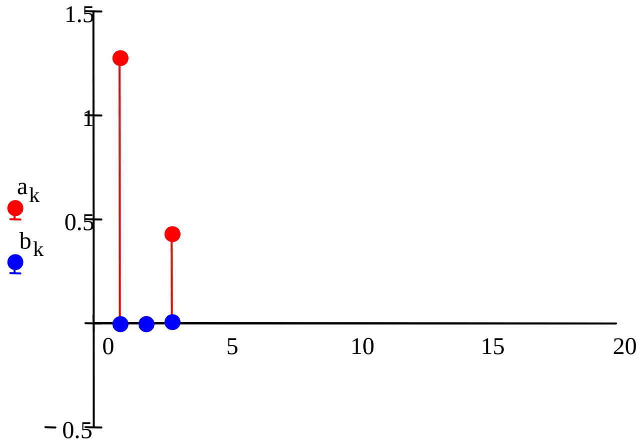
Spectrum Of A Few Common Signals

- Square wave



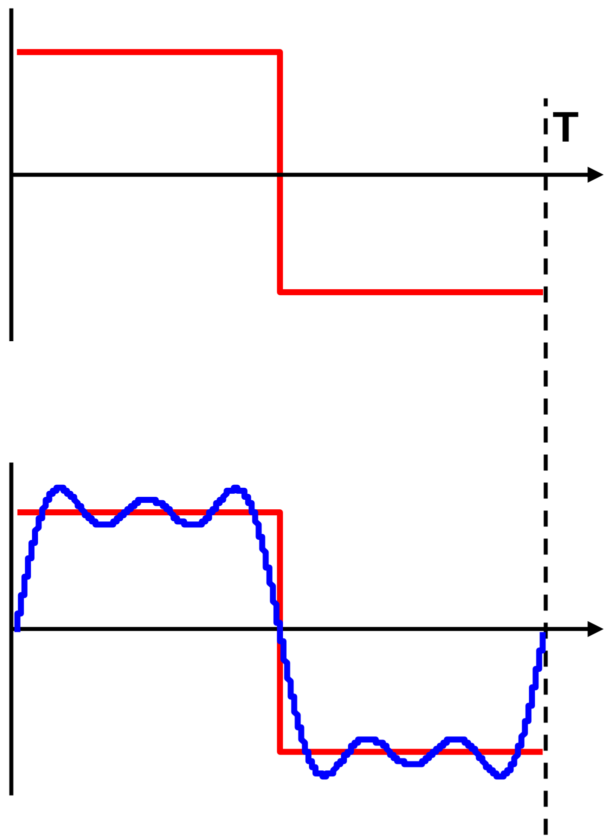


N = 3

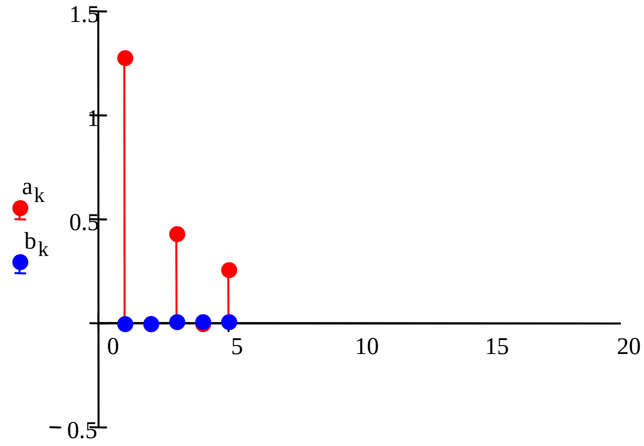


Spectrum Of A Few Common Signals

- Square wave

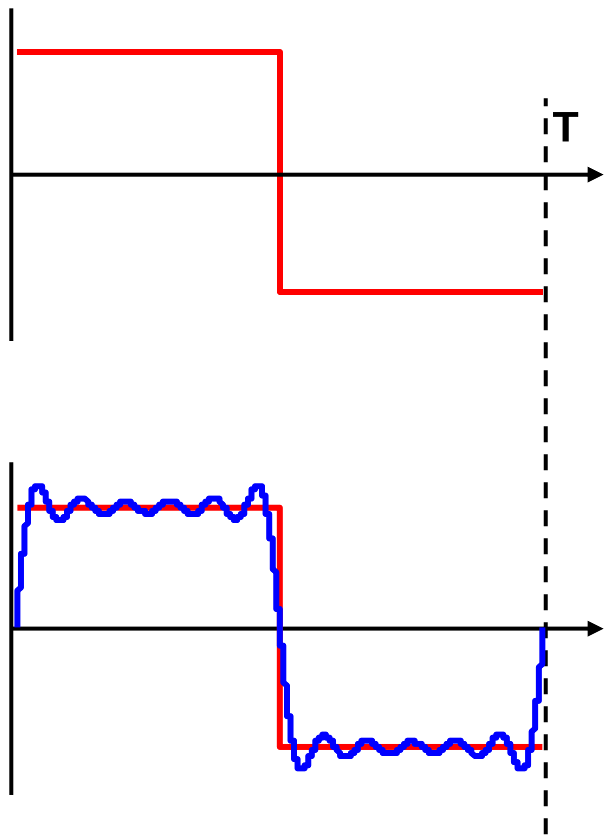


N = 5

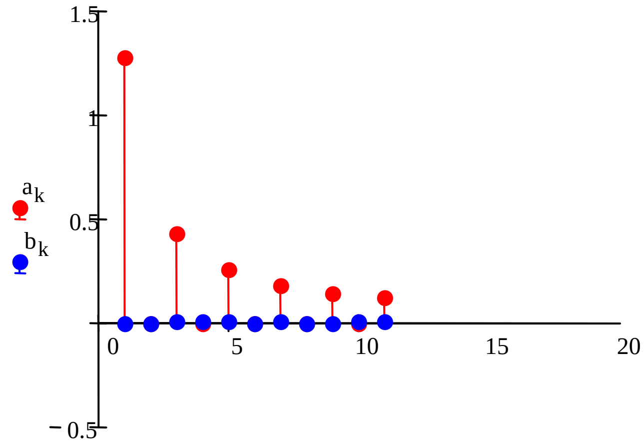


Spectrum Of A Few Common Signals

- Square wave

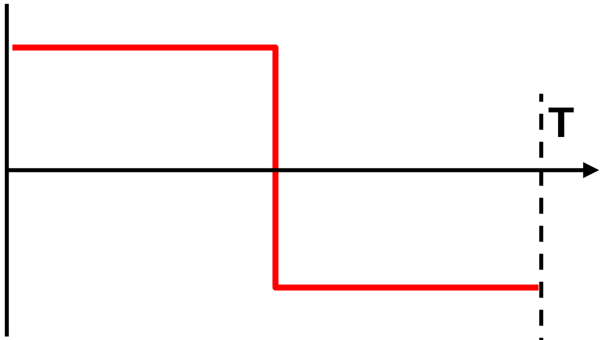


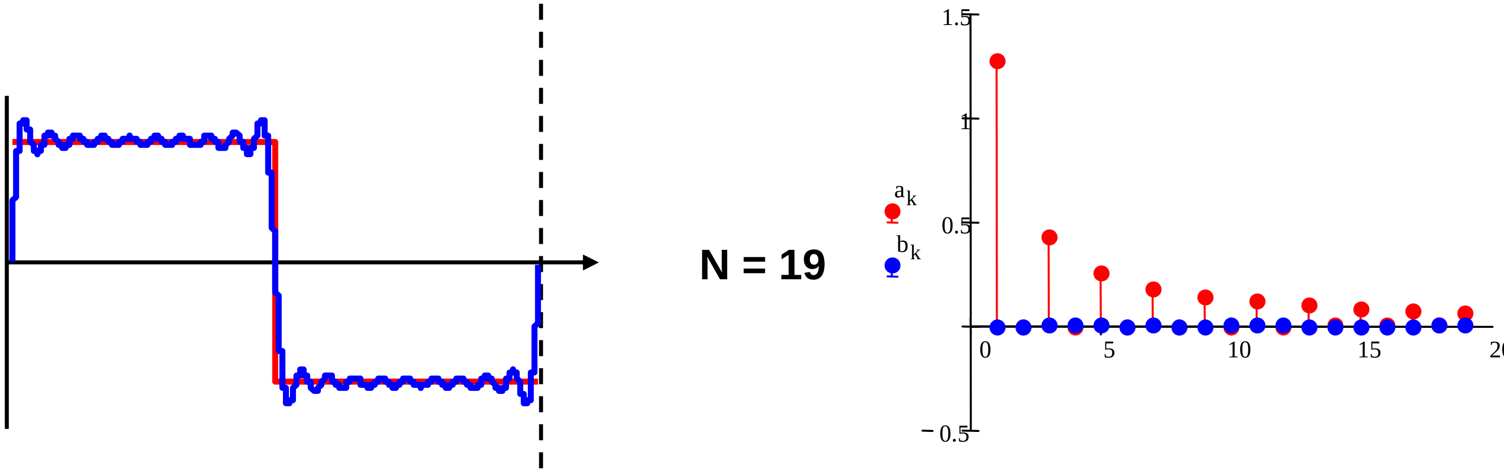
N = 11



Spectrum Of A Few Common Signals

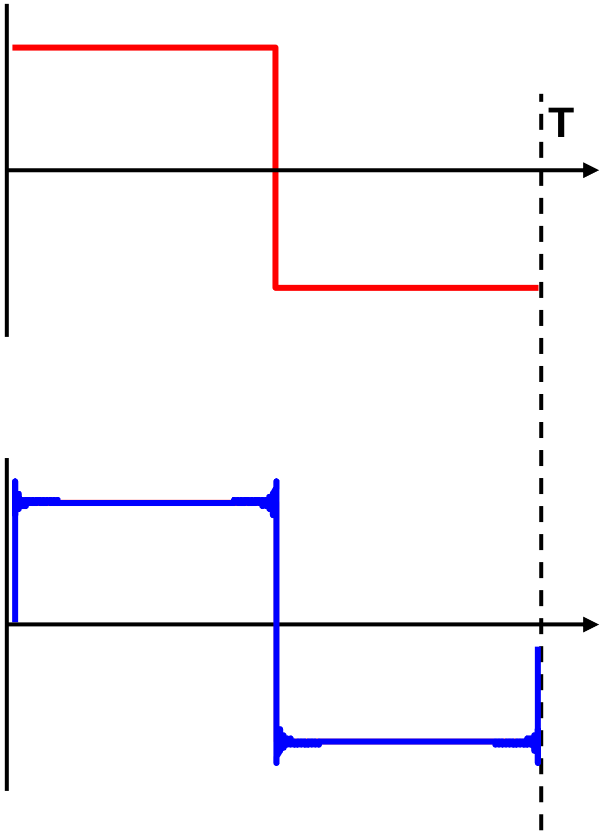
- Square wave



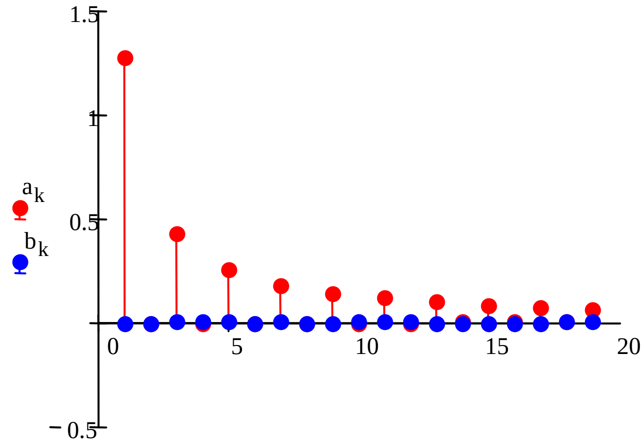


Spectrum Of A Few Common Signals

- Square wave

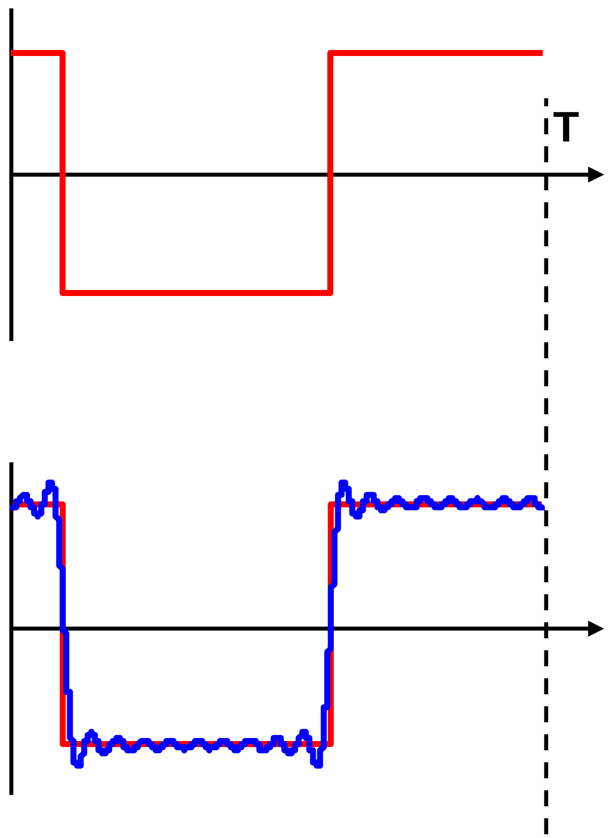


N = 200

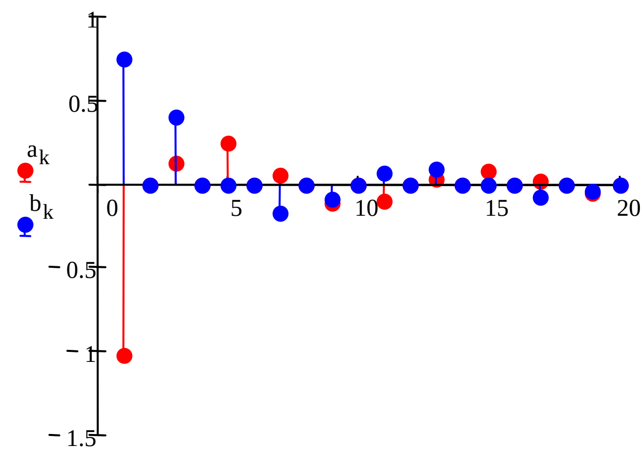


Spectrum Of A Few Common Signals

- Shifted square wave



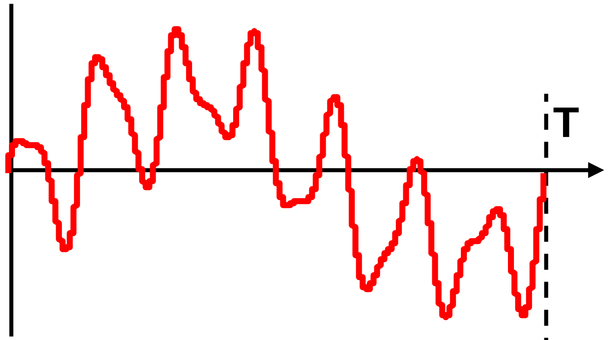
N = 19

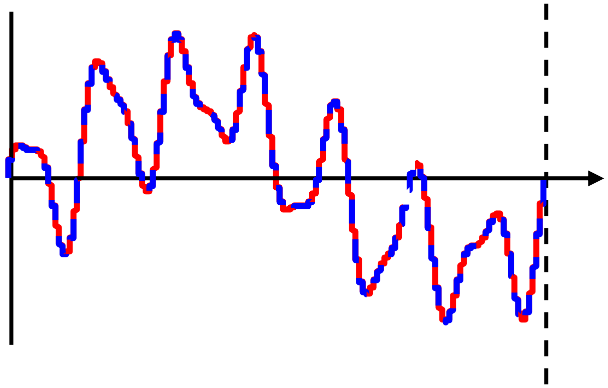


Spectrum Of A Few Common Signals

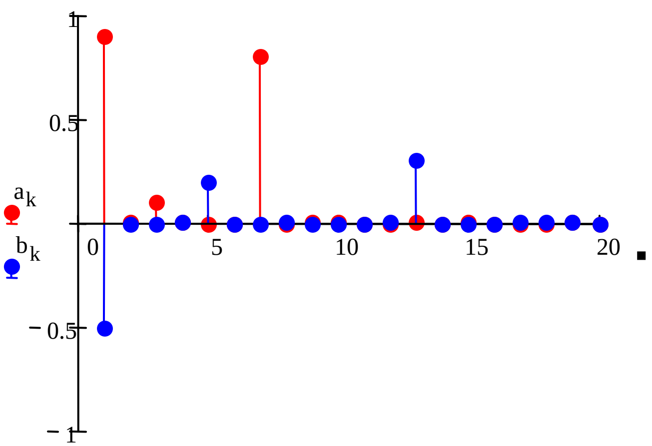
- Arbitrary sum of sinusoids

$$f(x) = .9\sin(2\pi x) - .5\cos(2\pi x) + .1\sin(3\cdot 2\pi x) + .2\cos(5\cdot 2\pi x) + .8\sin(7\cdot 2\pi x) + .3\cos(13\cdot 2\pi x)$$



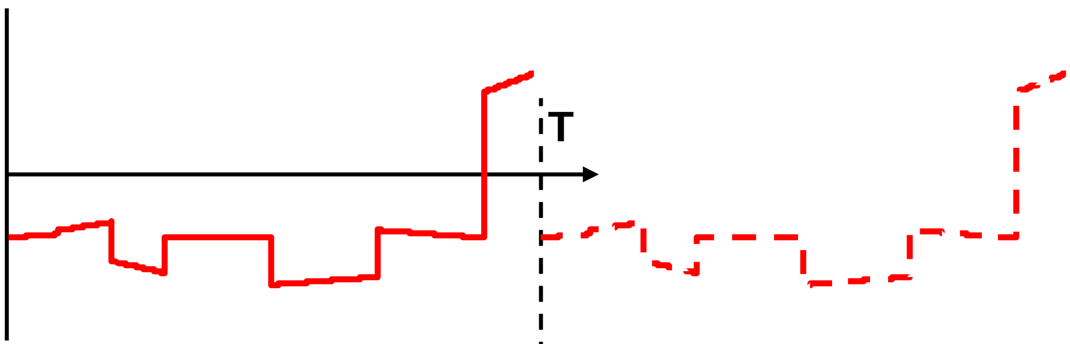


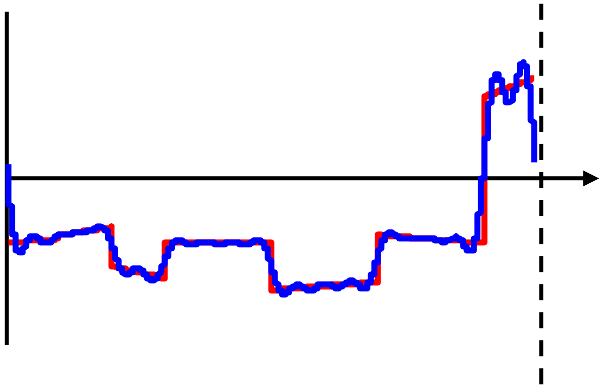
N = 19

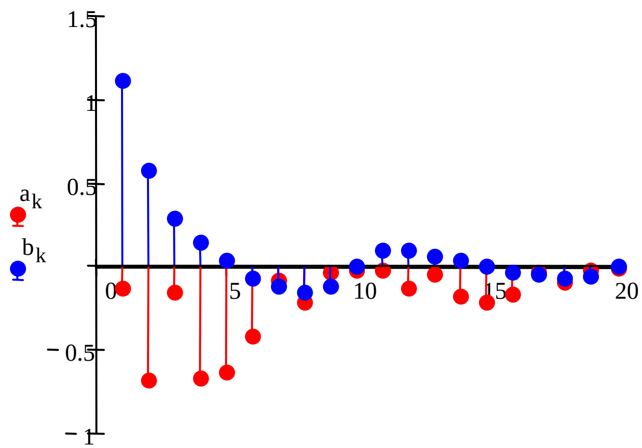


Spectrum Of Non-Periodic Signals

- Treat the signal as though it is periodic







Generalizing Fourier Series

$$f(t) = a_0 + \left(\sum_{i=1}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t \right) \right) + \left(\sum_{i=1}^{\infty} b_i \sin\left(i \frac{2\pi}{T} t \right) \right)$$

Fourier series

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(i \frac{2\pi}{T} t \right) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(i \frac{2\pi}{T} t \right) dt$$

Generalizing Fourier Series

$$f(t) = a_0 + \left(\sum_{i=1}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t \right) \right) + \left(\sum_{i=1}^{\infty} b_i \sin\left(i \frac{2\pi}{T} t \right) \right)$$

Fourier series

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(i \frac{2\pi}{T} t \right) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(i \frac{2\pi}{T} t \right) dt$$

Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Generalizing Fourier Series

$$f(t) = a_0 + \left(\sum_{i=1}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t \right) \right) + \left(\sum_{i=1}^{\infty} b_i \sin\left(i \frac{2\pi}{T} t \right) \right)$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

Fourier series

$$a_0 = \frac{1}{T} \int_0^T f(t)dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(i \frac{2\pi}{T} t \right) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(i \frac{2\pi}{T} t \right) dt$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi kt}{T}} dt$$

Euler's formula

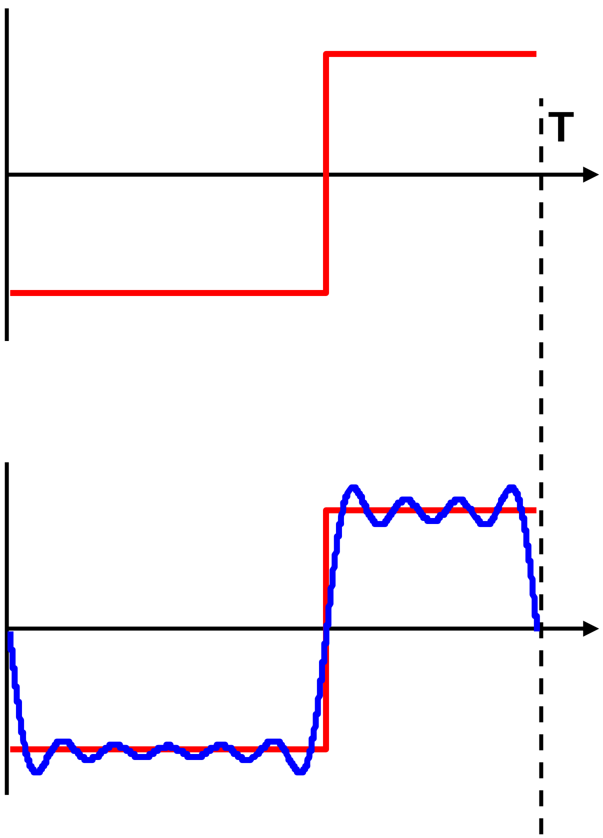
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

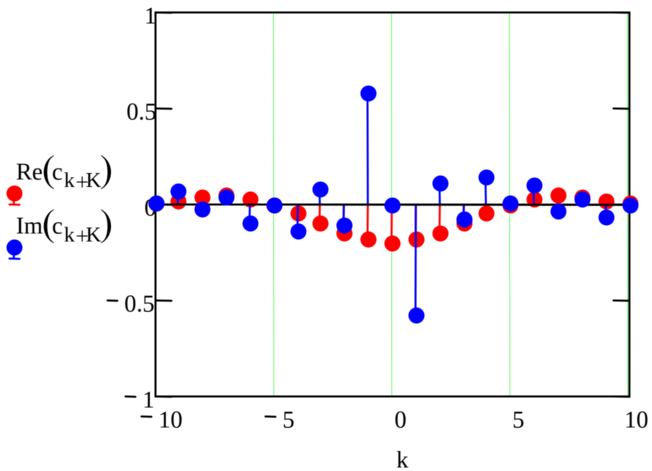
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex Spectrum Of A Signal

- Shifted square wave



N = 19



Generalizing The Fourier Series

- Start with the complex Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi kt}{T}} dt$$

Generalizing The Fourier Series

- Change variables

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Replace 2π/T with ω_0

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi kt}{T}} dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

Generalizing The Fourier Series: The Fourier Transform

- Consider what happens when the analysis period is allowed to increase

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Replace 2π/T with ω_0

Let ω_0 go to 0
T becomes infinite

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi kt}{T}} dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Applying The Fourier Transform To Sampled Signals

- Continuous samples of f(t), F(ω) are not available

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Applying The Fourier Transform To Sampled Signals

- Continuous samples of f(t), F(ω) are not available

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$f(n\Delta t) = \frac{1}{N}\sum_{k=0}^{N-1} F(k\Delta f)e^{j(2\pi k\Delta f)(n\Delta t)}$$

Discrete time

Discrete frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F(k\Delta f) = \sum_{n=0}^{N-1} f(n\Delta t)e^{-j(2\pi k\Delta f)(n\Delta t)}$$

Applying The Fourier Transform To Sampled Signals

- Continuous samples of f(t), F(ω) are not available

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$f(n\Delta t) = \frac{1}{N}\sum_{k=0}^{N-1} F(k\Delta f)e^{j(2\pi k\Delta f)(n\Delta t)}$$

Discrete time

Discrete frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F(k\Delta f) = \sum_{n=0}^{N-1} f(n\Delta t)e^{-j(2\pi k\Delta f)(n\Delta t)}$$

N-point Discrete Fourier Transform (DFT)

Applying The Fourier Transform To Sampled Signals

- Continuous samples of f(t), F(ω) are not available

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$f(n\Delta t) = \frac{1}{N}\sum_{k=0}^{N-1} F(k\Delta f)e^{j(2\pi k\Delta f)(n\Delta t)}$$

Discrete time

Note symmetry of e^{jx}

Not all N^2 factors need be calculated

Discrete frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F(k\Delta f) = \sum_{n=0}^{N-1} f(n\Delta t)e^{-j(2\pi k\Delta f)(n\Delta t)}$$

~~N-point Discrete Fourier Transform (DFT)~~

If $N=2^M$, (N x log(N)) operations needed for Fast Fourier Transform (FFT)

Next time

- Statistical Analysis of Experimental Data (read Ch. 6)