Design IV

E232 Fall 07

Class 14

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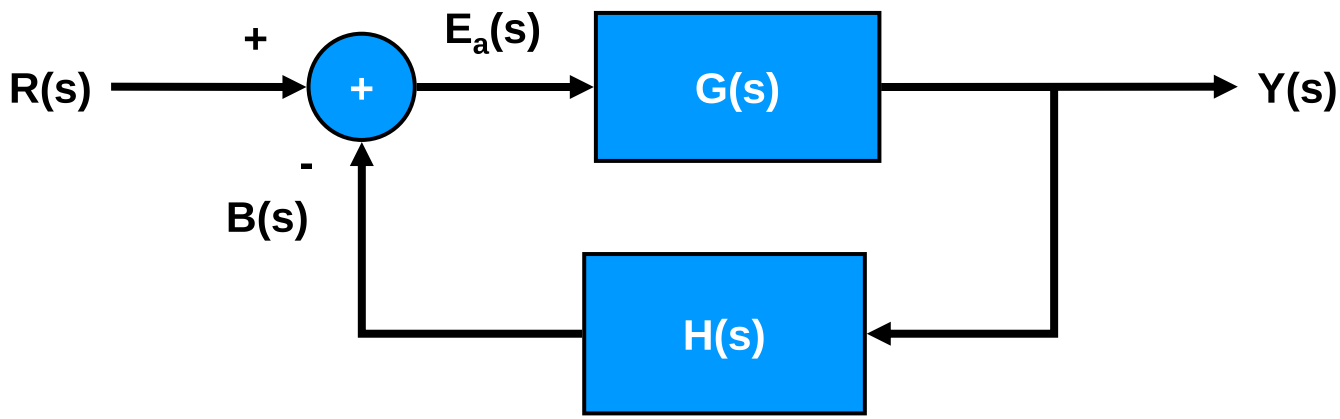
Transfer Function of a Feedback System

- Consider a generic feedback control system



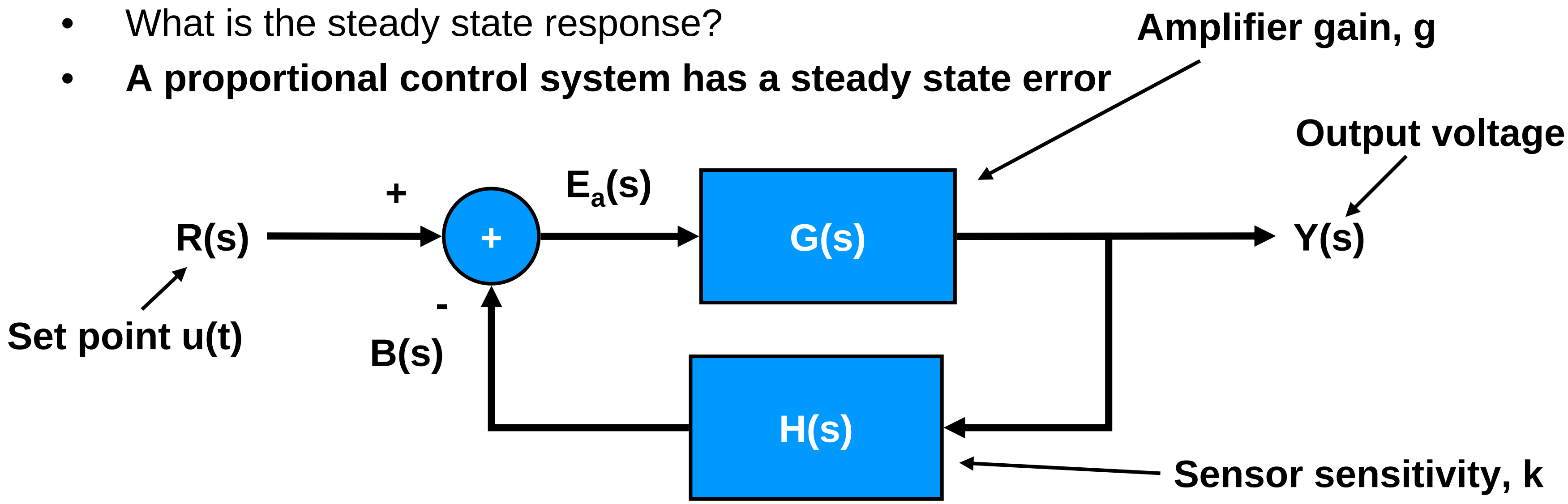
$$Y(s) = \big(R(s) - Y(s)H(s)\big)G(s)$$

$$Y(s)\left[1 + G(s)H(s)\right] = R(s)G(s)$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?
- **A proportional control system has a steady state error**



$$Y(s) = \frac{g}{1+gk}\frac{1}{s}$$

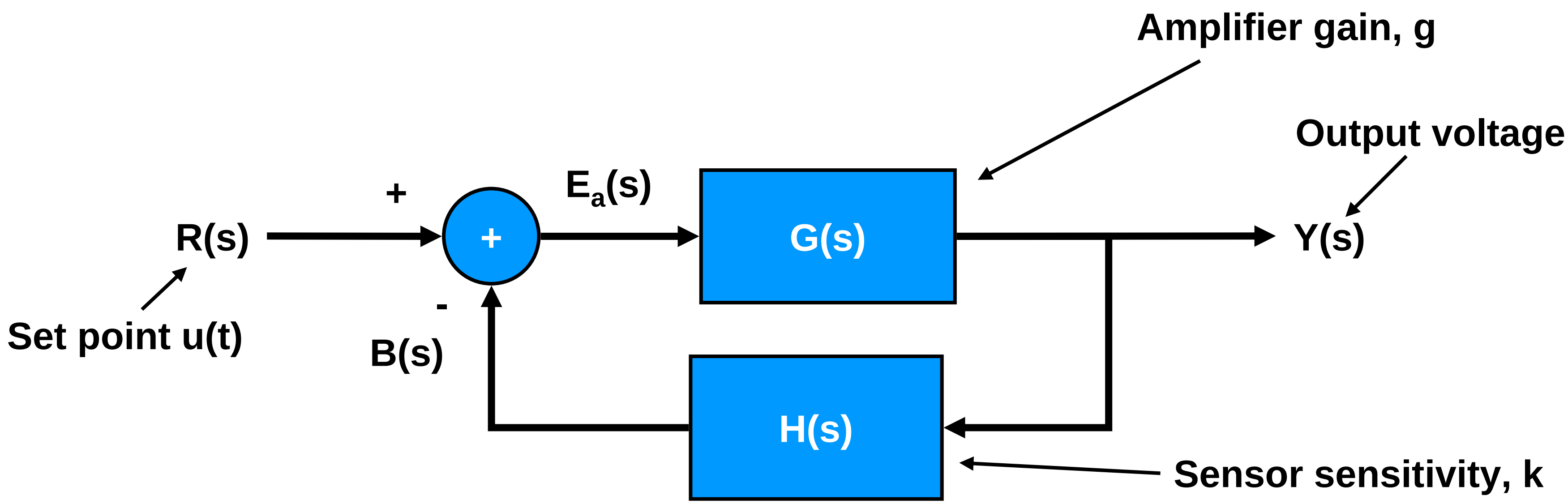
From the final value theorem:

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} sZ(s) \qquad \lim_{t\to\infty} e_a(t) = \lim_{s\to 0} s\frac{1}{s(1+gk)}$$

$$\boxed{\lim_{t\to\infty} e_a(t) = \frac{1}{1+gk}}$$

Steady State Response

- To reduce the steady state error:



But, what if error signal was something like:

$$\lim_{t \to \infty} e_a(t) = \lim_{s \to 0} s \frac{1}{s\left(1 + g\frac{k}{s}\right)}$$

Then:

$$\lim_{t \to \infty} e_a(t) = \lim_{s \to 0} \frac{1}{\left(1 + g\frac{k}{s}\right)} = \frac{s}{s + gk} = 0$$

Remember

$$\frac{1}{s} \equiv \int dt$$

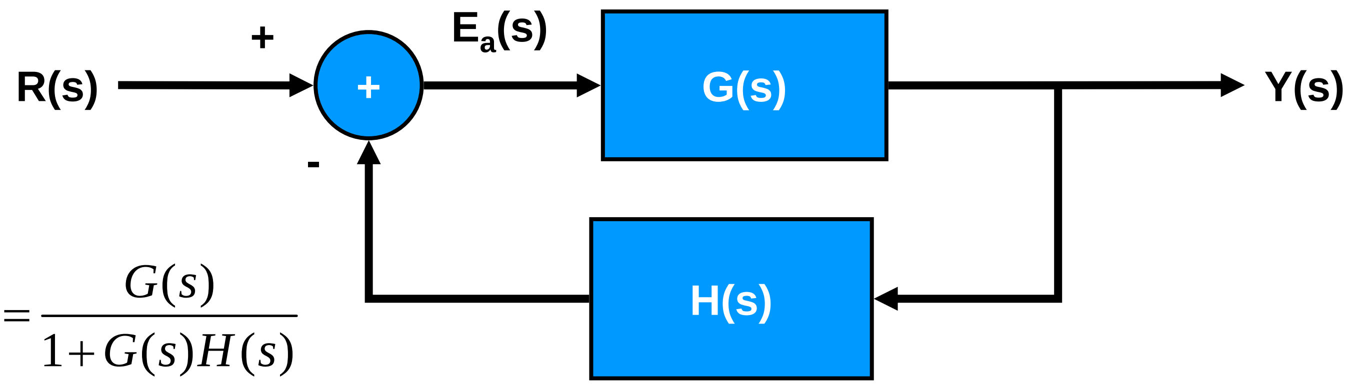
Integration of error signal allows steady state error to go to zero

Today’s topics

- Control systems
  - Unity feedback
  - Transient response
  - PID control
  - System types
  - Steady state error
  - Performance indices

Unity Feedback

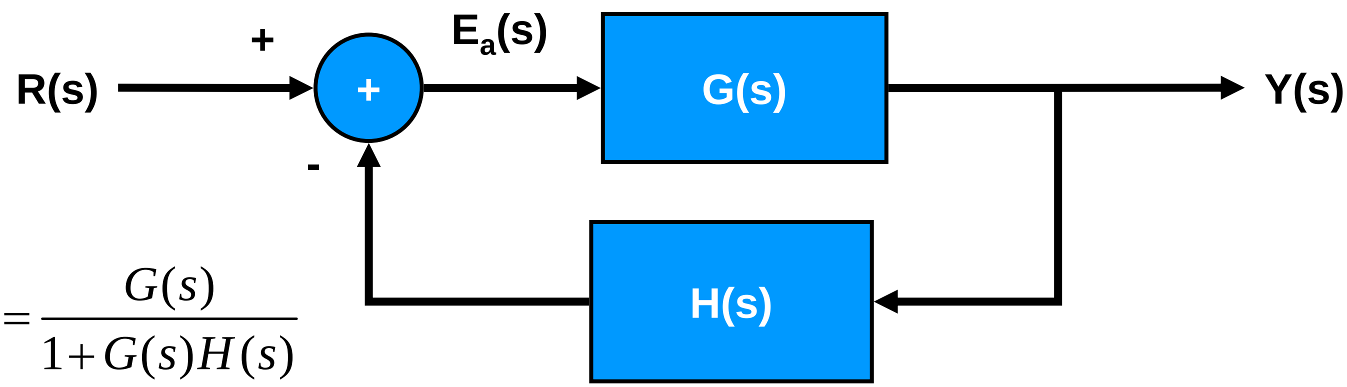
- Generic control system



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

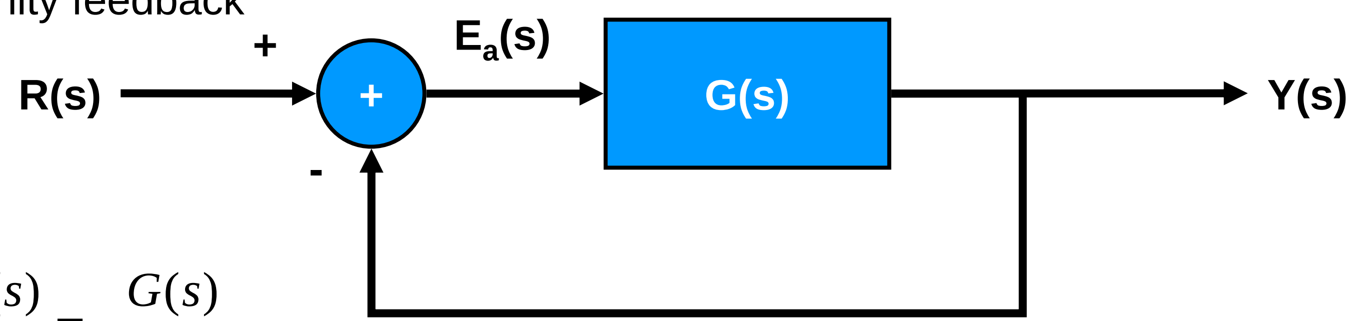
Unity Feedback

- Generic control system



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

- Unity feedback

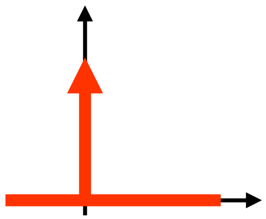


$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Transient Response

- Test signals

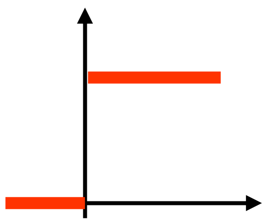
r(t) **R(s)**



Impulse

$$r_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases}$$

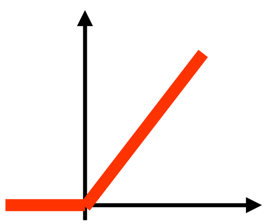
$$1$$



Step

$$r(t) = \begin{cases} A & t>0 \\ 0 & t<0 \end{cases}$$

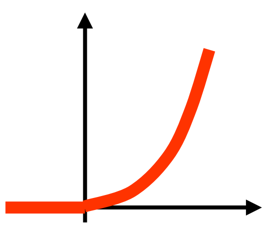
$$\frac{A}{s}$$



Ramp

$$r(t) = \begin{cases} At & t>0 \\ 0 & t<0 \end{cases}$$

$$\frac{A}{s^2}$$



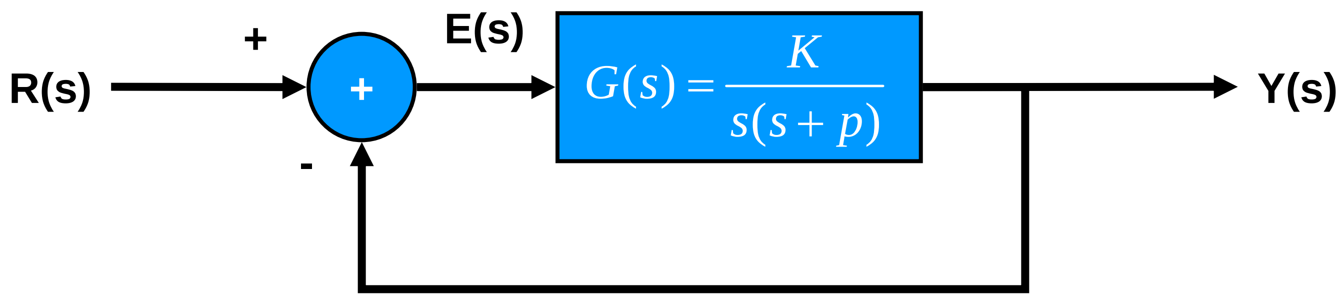
Parabolic

$$r(t) = \begin{cases} At^2 & t>0 \\ 0 & t<0 \end{cases}$$

$$\frac{2A}{s^3}$$

Second Order System Performance

- Consider the simple control system:



Second Order System Performance

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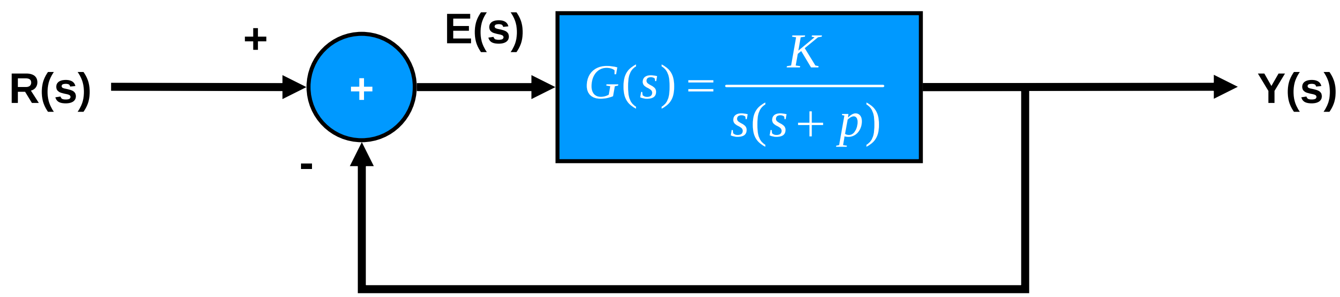
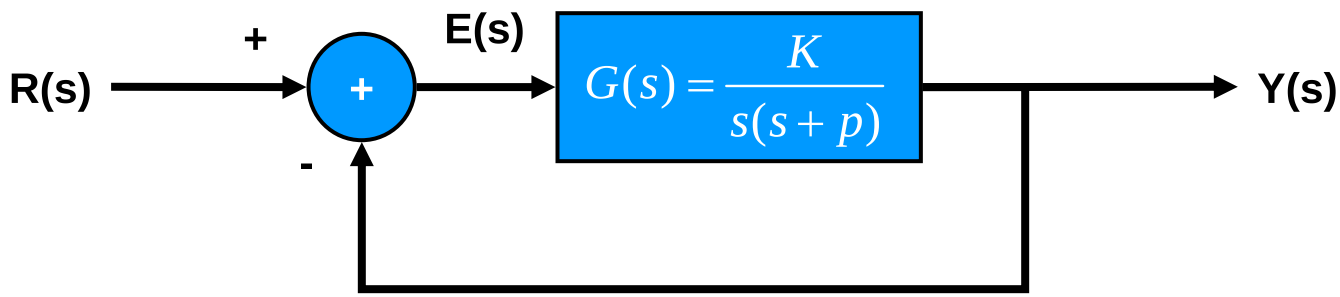


$$Y(s)=\frac{G(s)}{1+G(s)}R(s)=\frac{K}{s^2+ps+K}R(s)$$

Second Order System Performance

- Consider the simple control system:

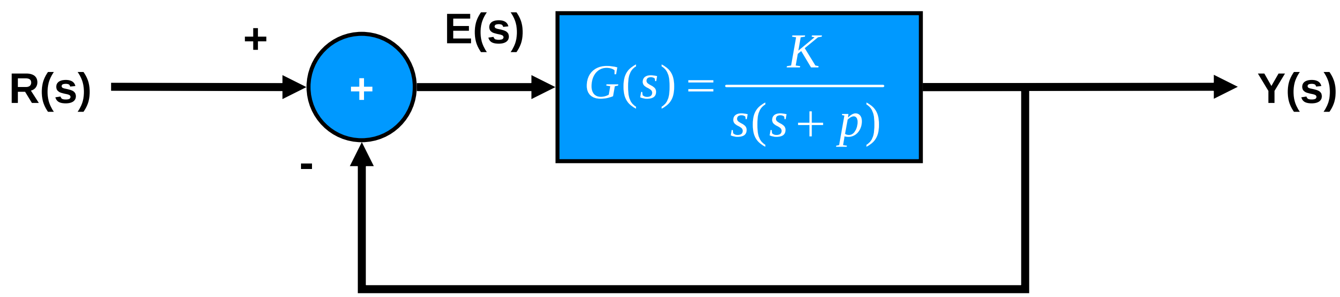


$$Y(s)=\frac{G(s)}{1+G(s)}R(s)=\frac{K}{s^2+ps+K}R(s)$$

$$Y(s)=\frac{\omega_n^{\ 2}}{\left(s^2+2\zeta\omega_n s+\omega_n^{\ 2}\right)}R(s)$$

Second Order System Performance

- Consider the simple control system:



$$Y(s) = \frac{G(s)}{1+G(s)} R(s) = \frac{K}{s^2 + ps + K} R(s)$$

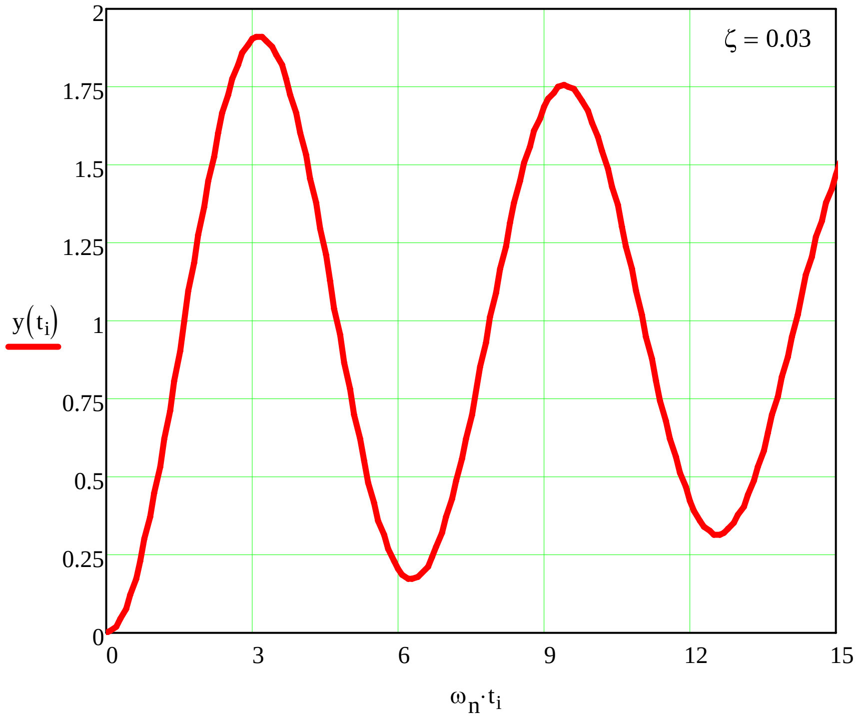
$$Y(s) = \frac{\omega_n^{\ 2}}{\left(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\right)} R(s)$$

With r(t) = u(t):

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}(\zeta)\right)$$

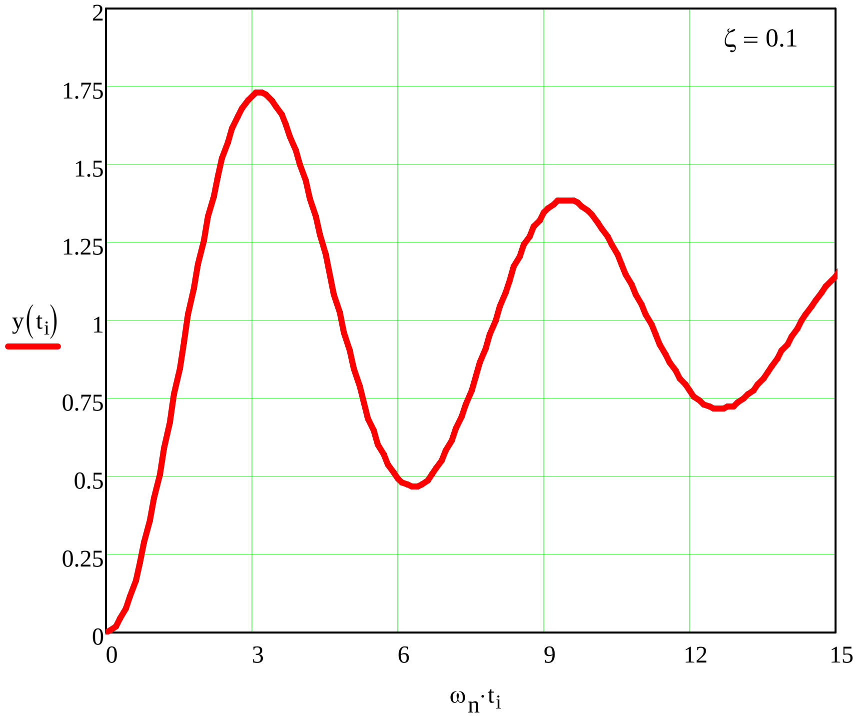
Second Order System Performance

- Effect of damping factor, ζ



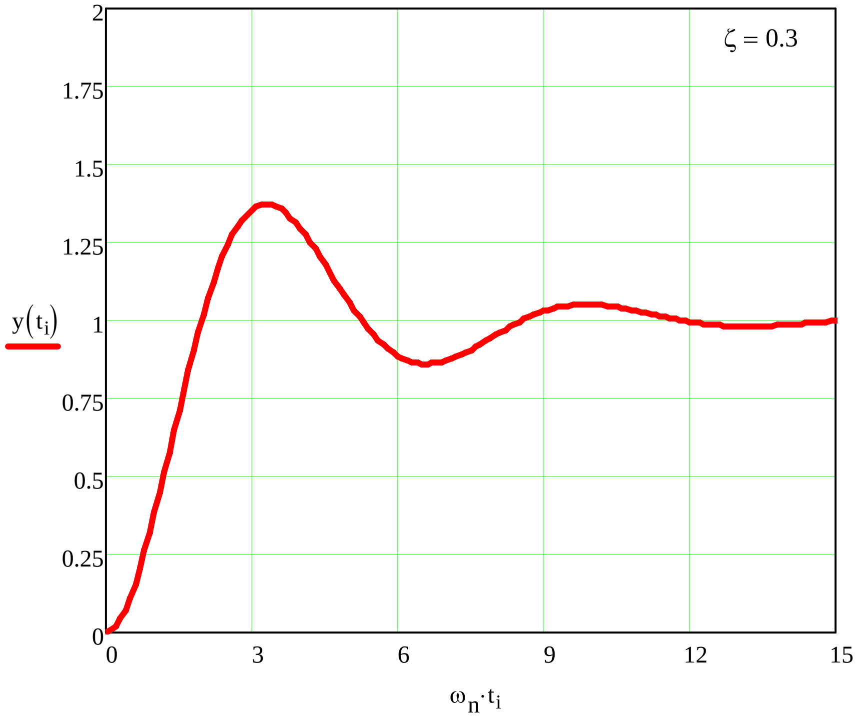
Second Order System Performance

- Effect of damping factor, ζ



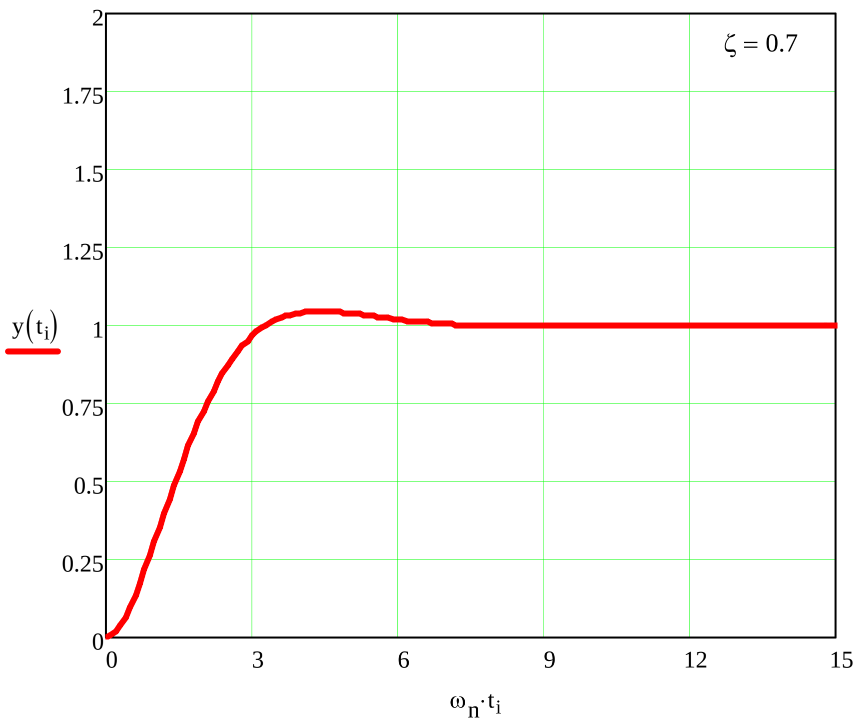
Second Order System Performance

- Effect of damping factor, ζ



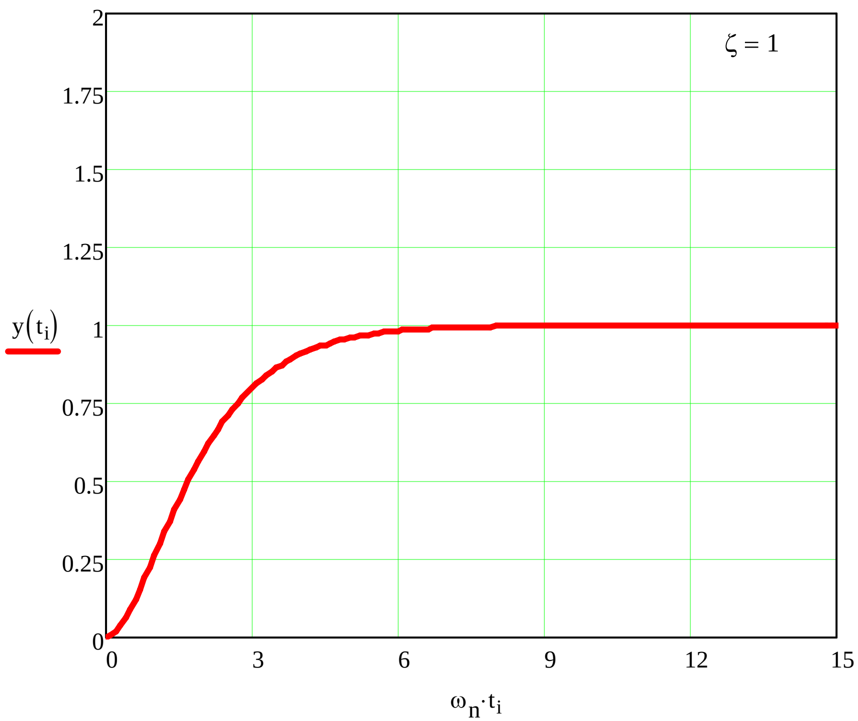
Second Order System Performance

- Effect of damping factor, ζ



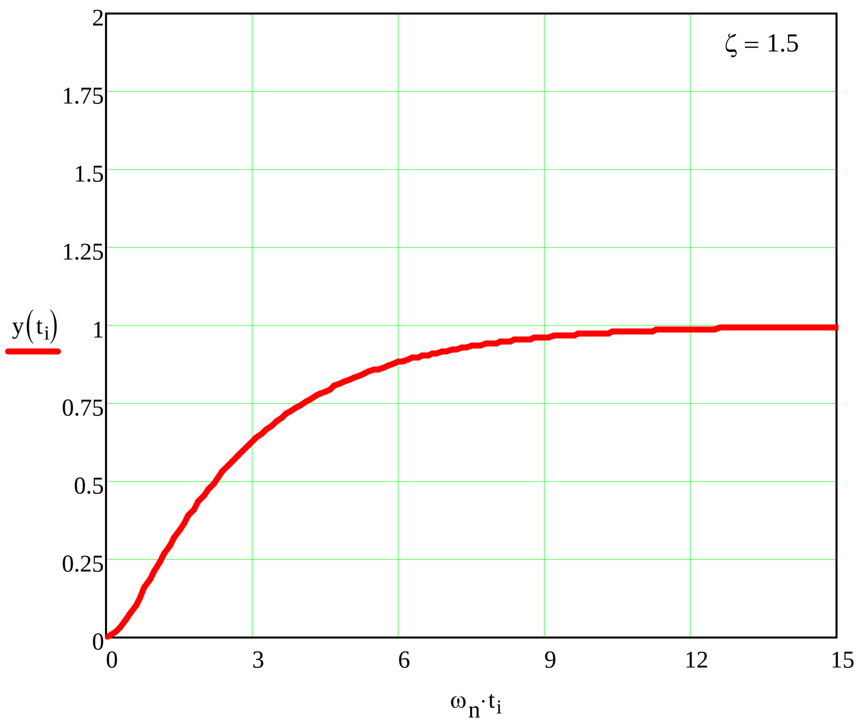
Second Order System Performance

- Effect of damping factor, ζ



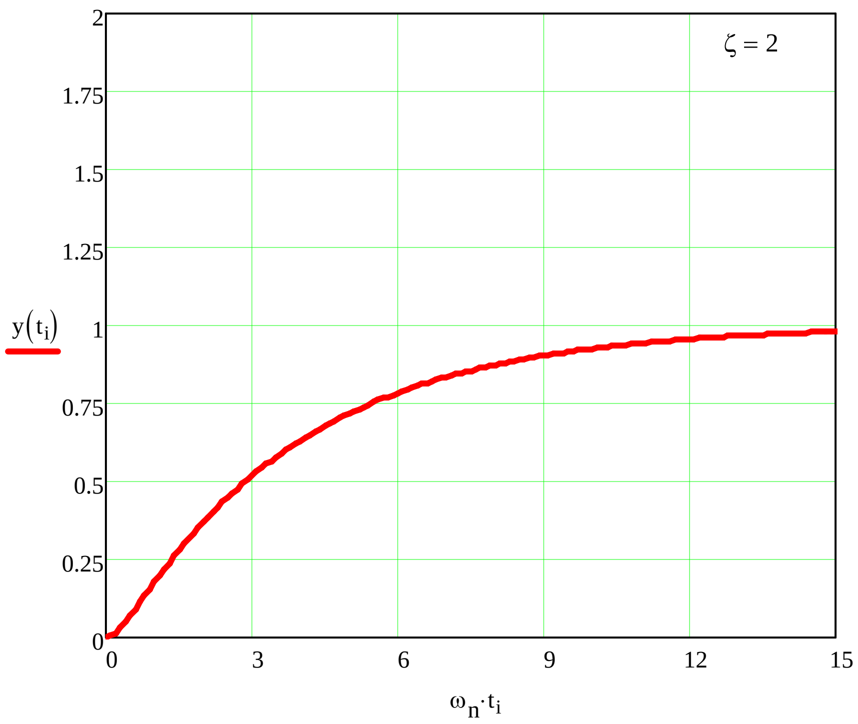
Second Order System Performance

- Effect of damping factor, ζ



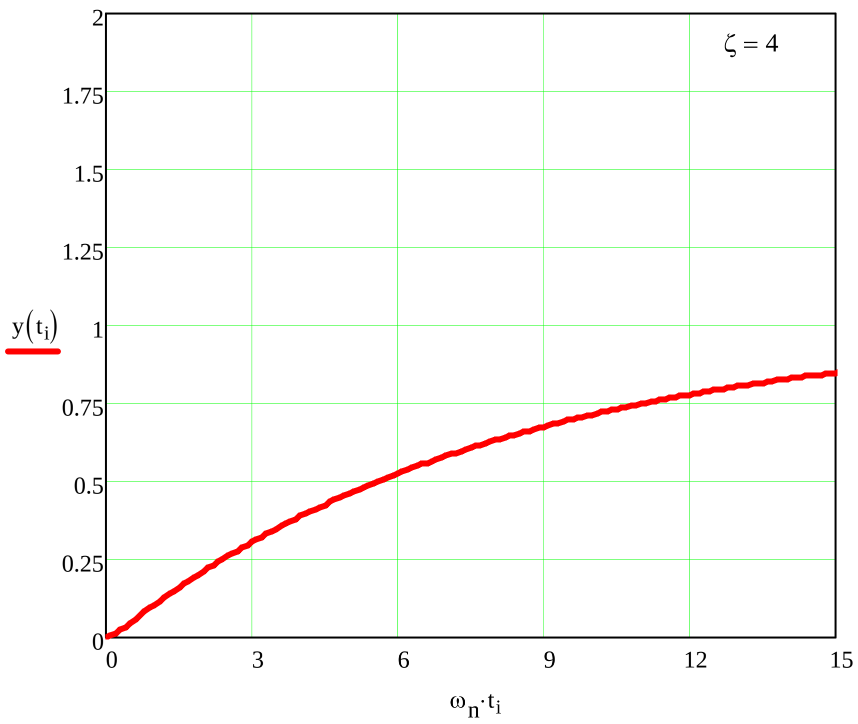
Second Order System Performance

- Effect of damping factor, ζ



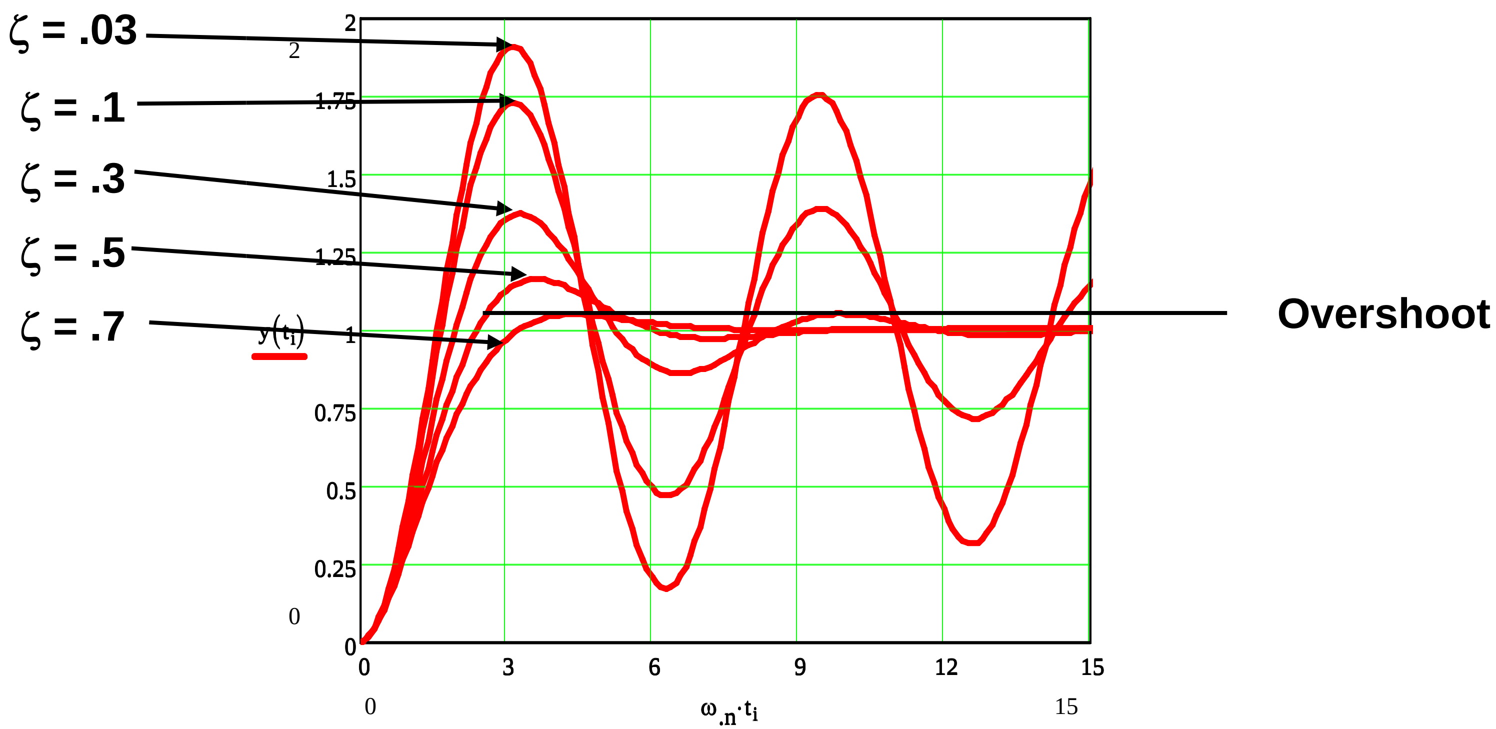
Second Order System Performance

- Effect of damping factor, ζ



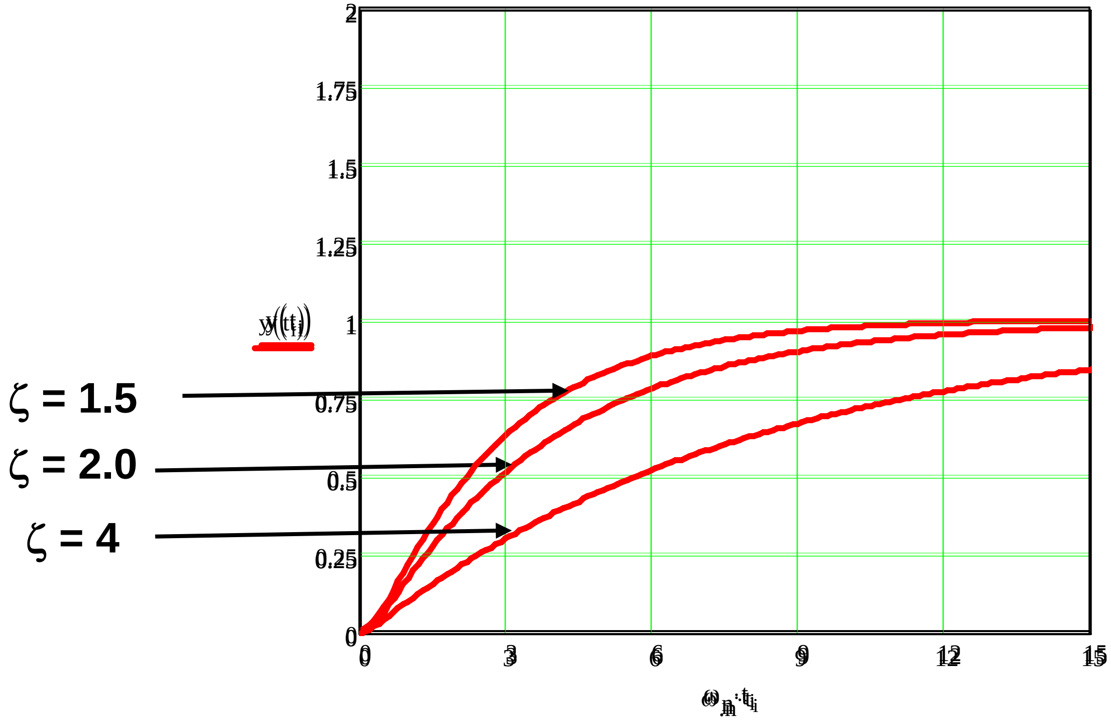
Second Order System Performance

- Under damped, $\zeta < 1$



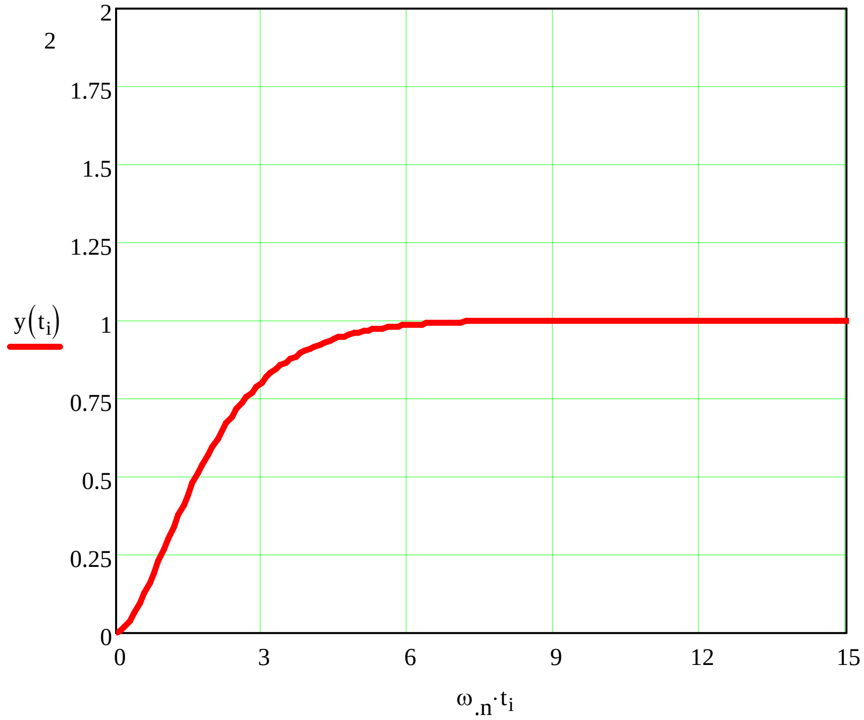
Second Order System Performance

- Over damped, $\zeta > 1$



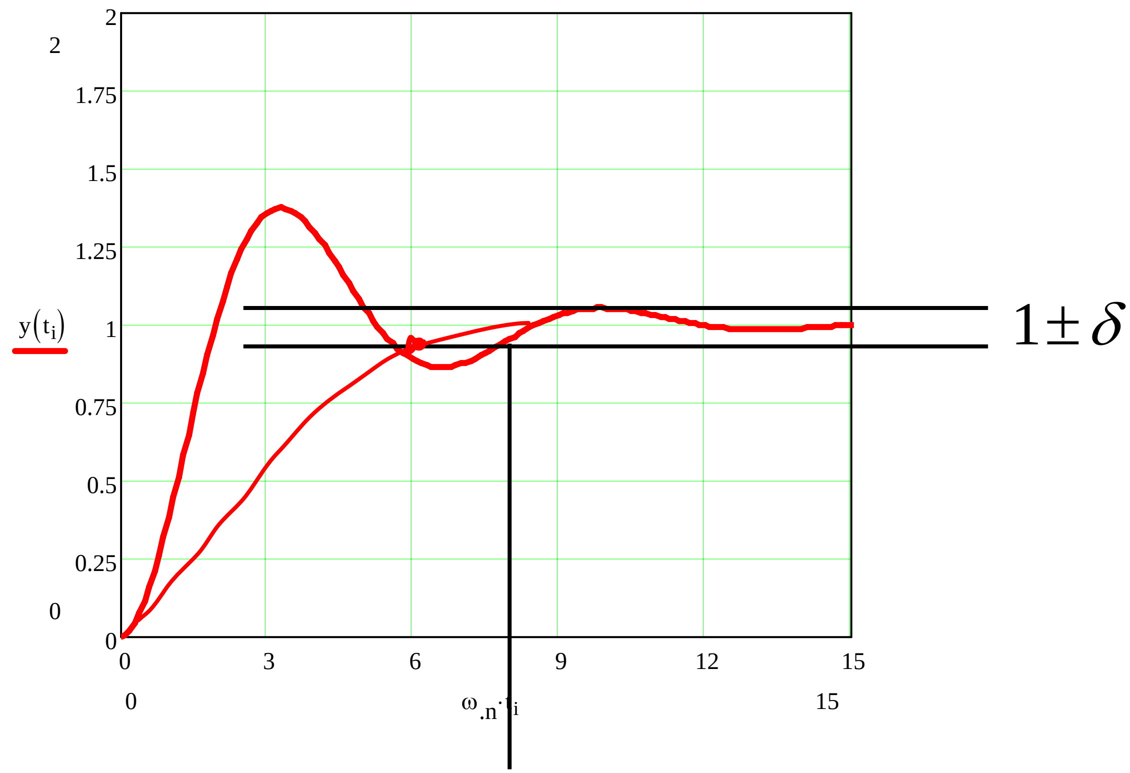
Second Order System Performance

- Critically damped, $\zeta = 1$



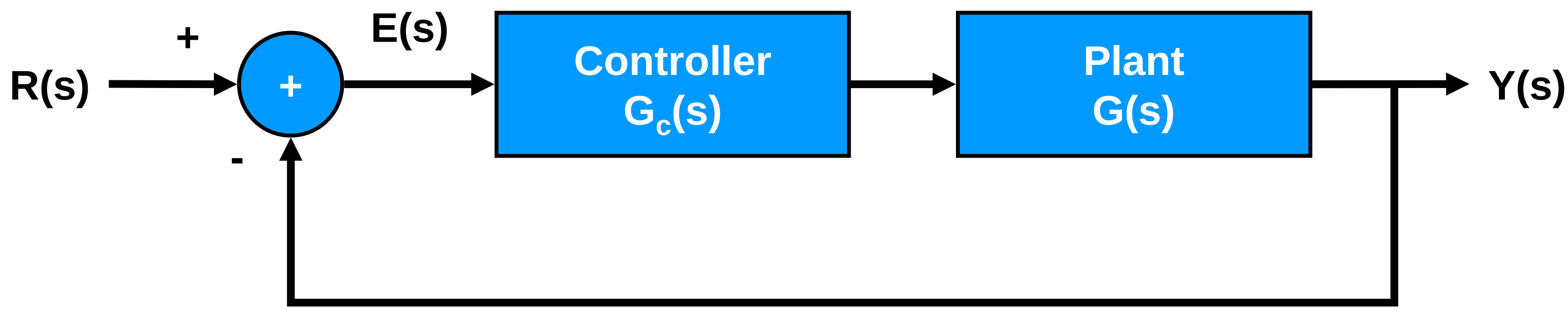
Second Order System Performance

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}(\zeta)\right)$$



Settling time $e^{-\zeta\omega_n T_s} < \delta$

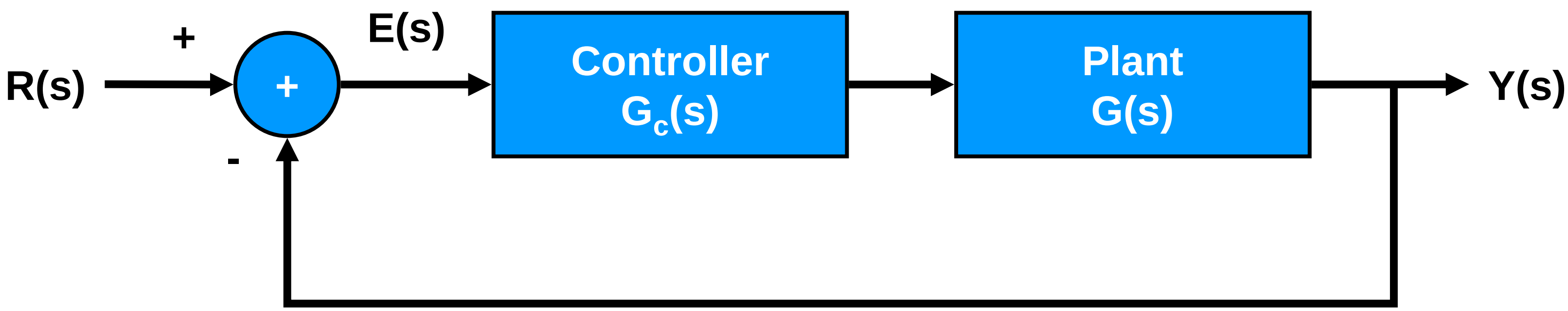
Proportional, Integral, Derivative (PID) Control



$$G_c(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_d s + 1}$$

$$G_c(s) \approx K_P + \frac{K_I}{s} + K_D s$$

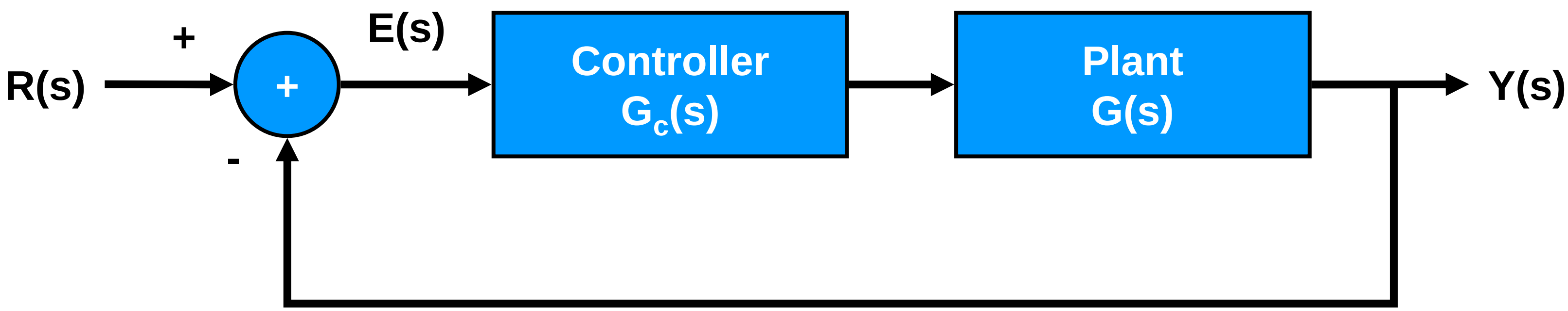
Proportional Plus Integral (PI) Control



$$G_c(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_d s + 1}$$

$$G_c(s) \approx K_P + \frac{K_I}{s} + \cancel{K_D s}$$

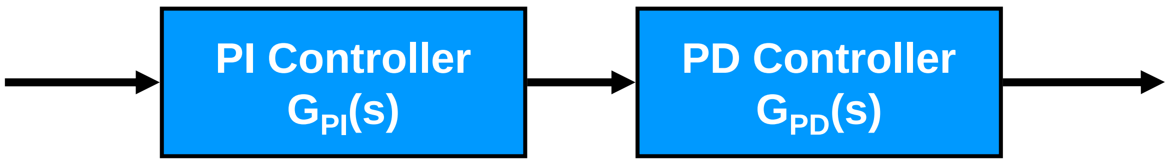
Proportional Plus Derivative (PD) Control



$$G_c(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau_d s + 1}$$

$$G_c(s) \approx K_P + \cancel{\frac{K_I}{s}} + K_D s$$

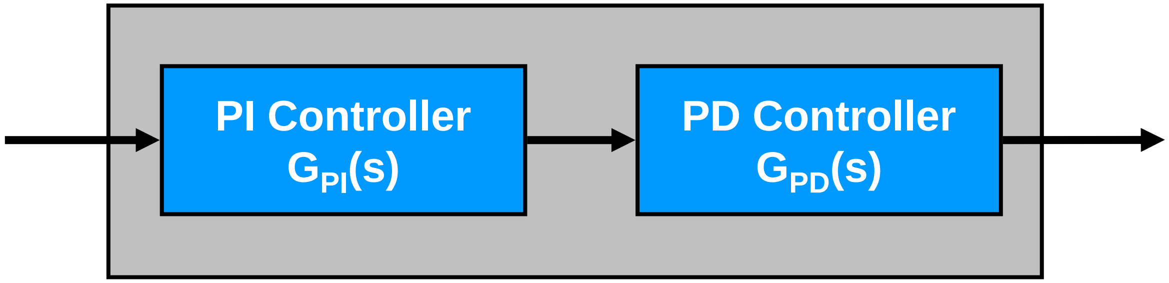
PID Control = PD + PI



$$G_{PI}(s) = K'_P + \frac{K'_I}{s}$$

$$G_{PD}(s) = K''_P + K''_D\, s$$

PID Control = PD + PI

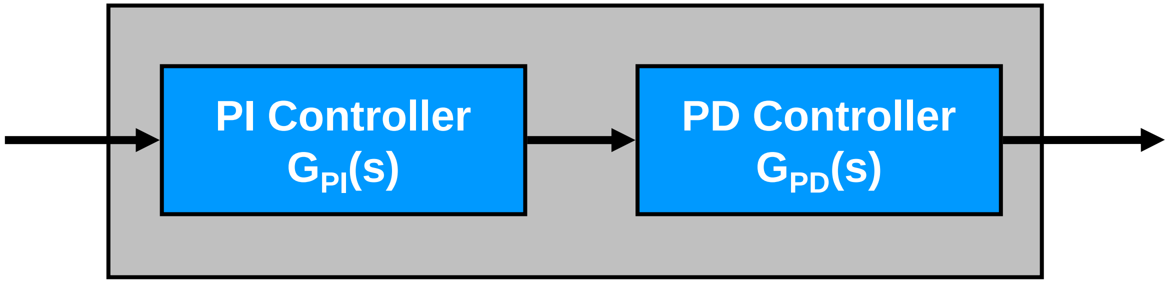


$$G_{PI}(s) = K'_P + \frac{K'_I}{s}$$

$$G_{PD}(s) = K''_P + K''_D\, s$$

$$G_c(s) = \left(K''_P + K''_D\, s\right)\left(K'_P + \frac{K'_I}{s}\right)$$

PID Control = PD + PI



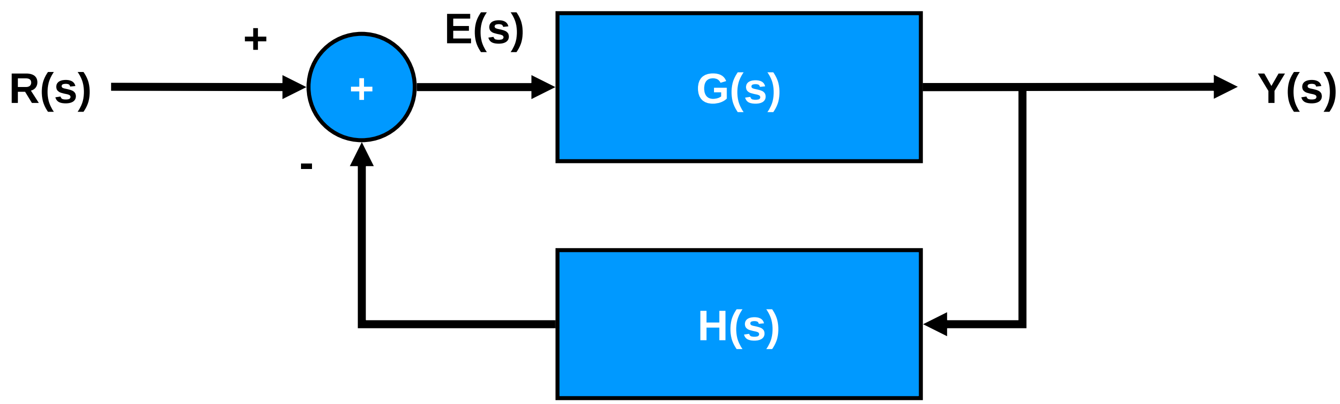
$$G_{PI}(s) = K'_P + \frac{K'_I}{s}$$

$$G_{PD}(s) = K''_P + K''_D\, s$$

$$G_c(s) = \left(K''_P + K''_D\, s\right)\left(K'_P + \frac{K'_I}{s}\right)$$

$$G_c(s) = \left(K'_P\, K''_P + K'_I\, K''_D\right) + \frac{K'_I\, K''_P}{s} + K'_P\, K''_D\, s$$

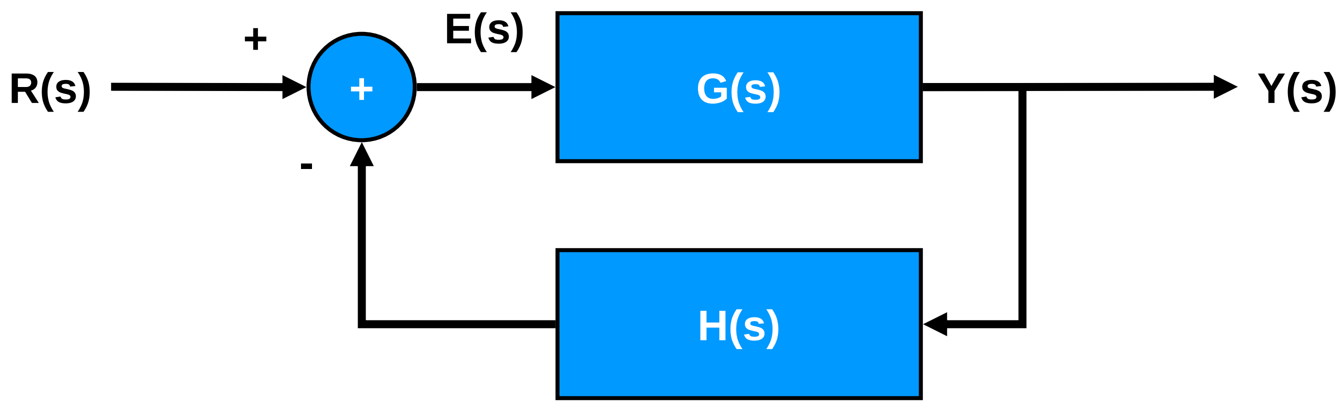
Type 0, Type 1, Type 2,… Systems



$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)...(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)...(T_p s+1)}$$

N is the number of integrations in the open loop transfer function

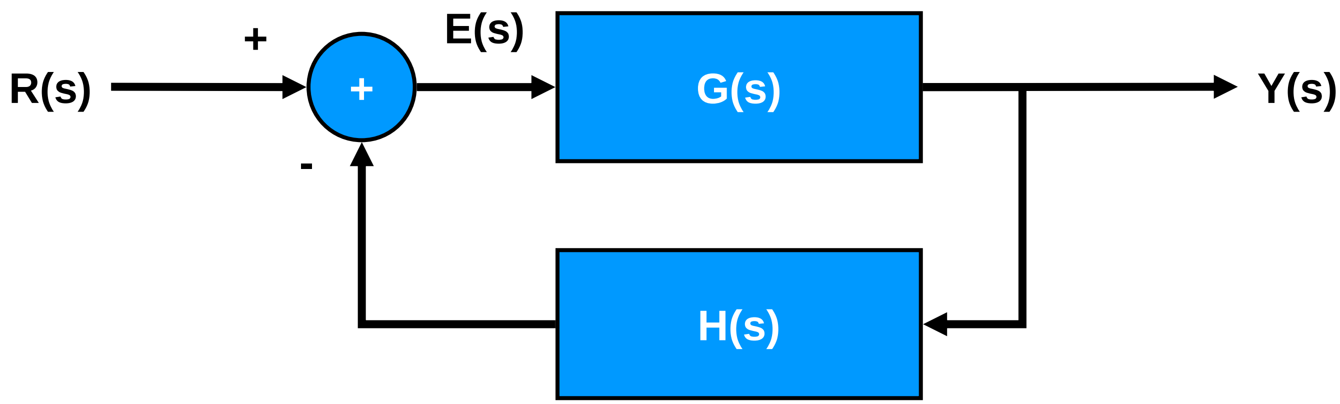
Type 0, Type 1, Type 2,… Systems



$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots(T_p s+1)}$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad \frac{E(s)}{R(s)} = 1 - \frac{Y(s)H(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

Type 0, Type 1, Type 2,… Systems

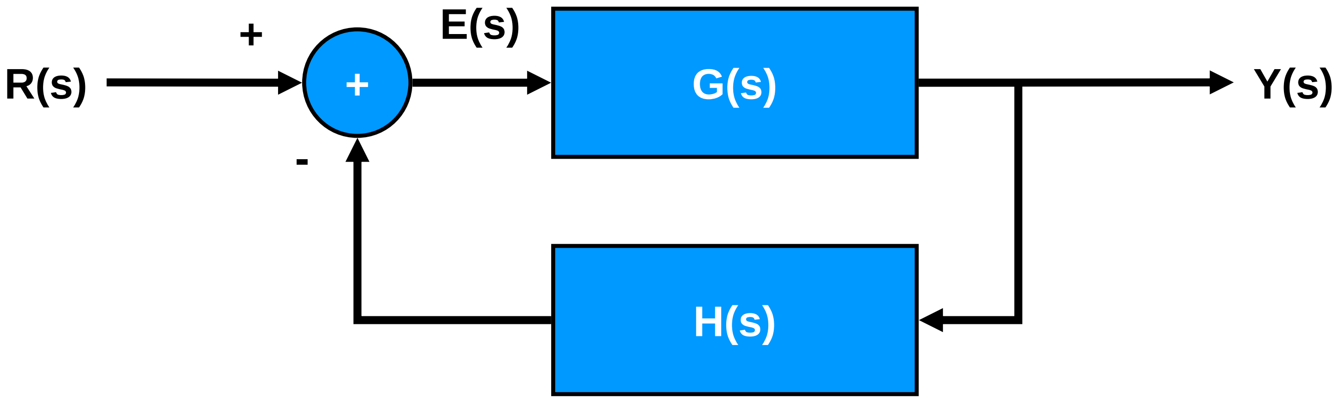


$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots(T_p s+1)}$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad \frac{E(s)}{R(s)} = 1 - \frac{Y(s)H(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

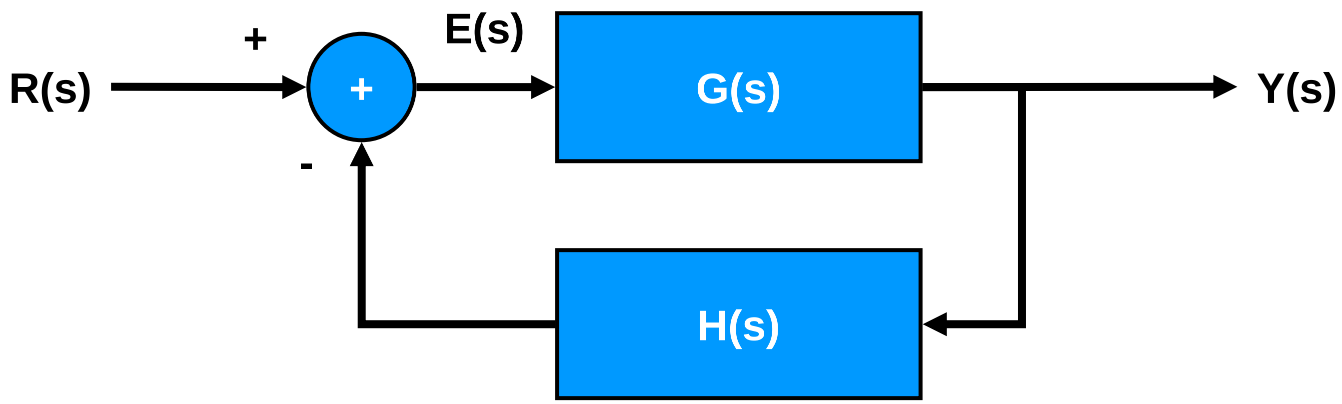
Type 0, Type 1, Type 2,… Systems



$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \frac{sR(s)}{1 + G(s)H(s)}$$

Type 0, Type 1, Type 2,… Systems



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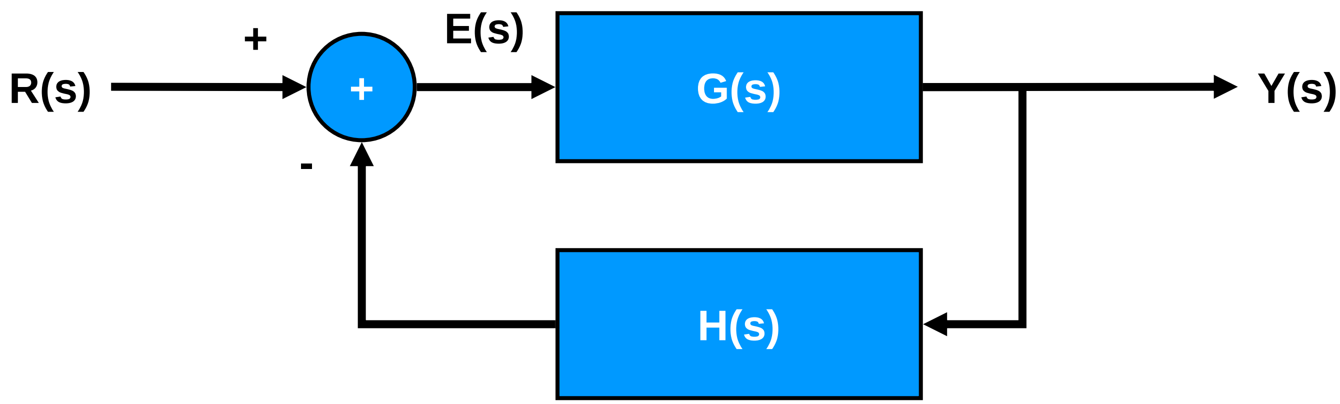
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \frac{sR(s)}{1 + G(s)H(s)}$$

Static position error coefficient:

$$K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

Type 0, Type 1, Type 2,… Systems

- With unit step input $$r(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \frac{s}{1 + G(s)H(s)} \frac{1}{s}$$

Static position error coefficient:

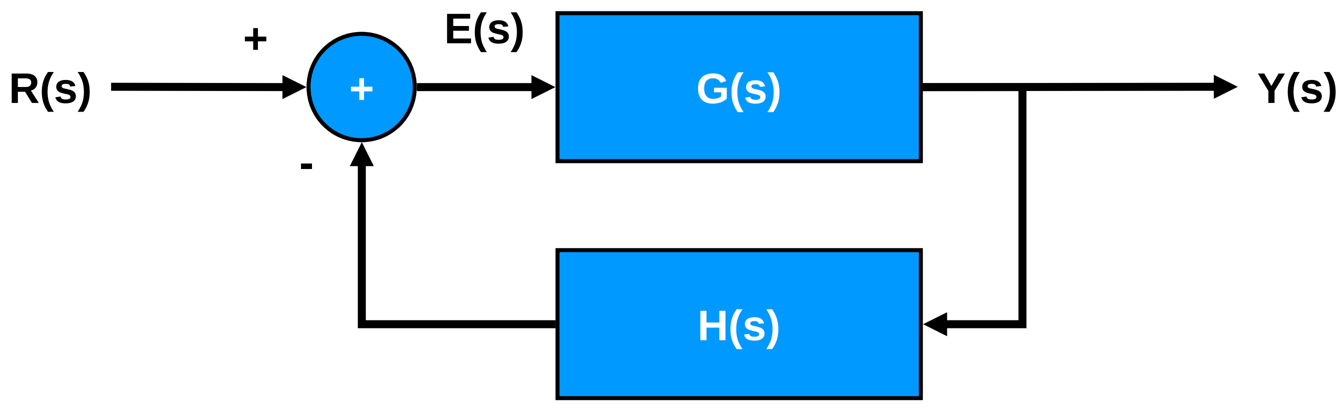
$$K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

For a unit step input:

$$e_{ss} = \frac{1}{1 + K_p}$$

Type 0, Type 1, Type 2,… Systems

- With unit step input $$r(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

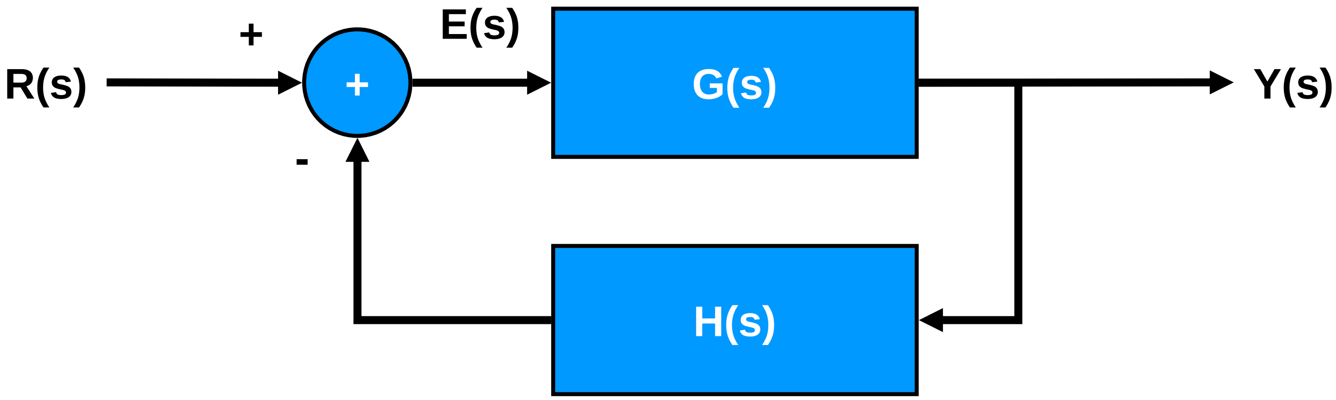


Type 0: $$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\ldots(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)\ldots(T_p s + 1)} = K \qquad e_{ss} = \frac{1}{1+K} > 0$$

Type 1 or higher: $$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)\ldots(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)\ldots(T_p s + 1)} \to \infty \qquad e_{ss} = 0$$

Type 0, Type 1, Type 2,… Systems

- With unit ramp input $$r(t)=\begin{cases}0 & t<0\\ t & t\geq 0\end{cases}$$



$$E(s)=\frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\frac{s}{1+G(s)H(s)}\frac{1}{s^2}$$

Static velocity error coefficient:

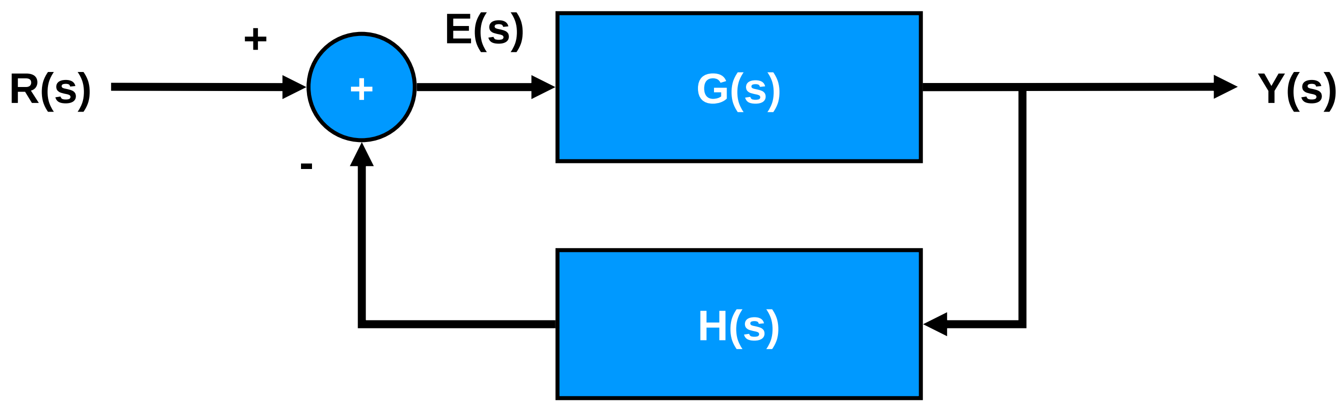
$$K_v=\lim_{s\to 0}sG(s)H(s)$$

For a unit ramp input:

$$e_{ss}=\frac{1}{K_v}$$

Type 0, Type 1, Type 2,… Systems

- With unit ramp input $r(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$



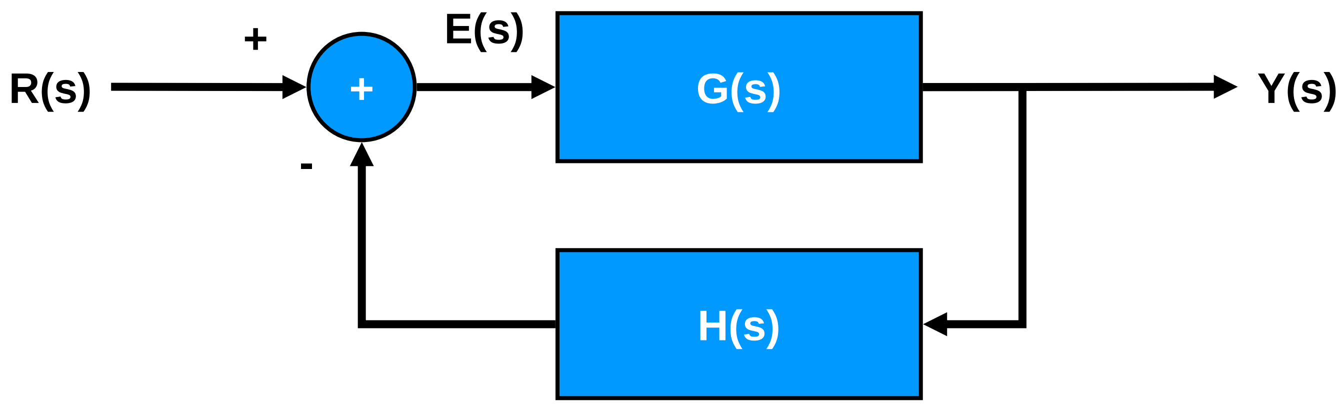
Type 0: $K_v = \lim_{s \to 0} \frac{sK(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{(T_1 s+1)(T_2 s+1)\ldots(T_p s+1)} = 0$ $\qquad e_{ss} = \frac{1}{K_v} \to \infty$

Type 1: $K_p = \lim_{s \to 0} \frac{sK(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{s(T_1 s+1)(T_2 s+1)\ldots(T_p s+1)} = K$ $\qquad e_{ss} = \frac{1}{K_v} = \frac{1}{K}$

Type 2 or higher: $K_v = \lim_{s \to 0} \frac{sK(T_a s+1)(T_b s+1)\ldots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\ldots(T_p s+1)} \to \infty$ $\qquad e_{ss} = \frac{1}{K_v} = 0$

Type 0, Type 1, Type 2,… Systems

- With unit parabolic input $r(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & t \geq 0 \end{cases}$



$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{s}{1+G(s)H(s)}\frac{1}{s^3}$$

Static acceleration error coefficient:

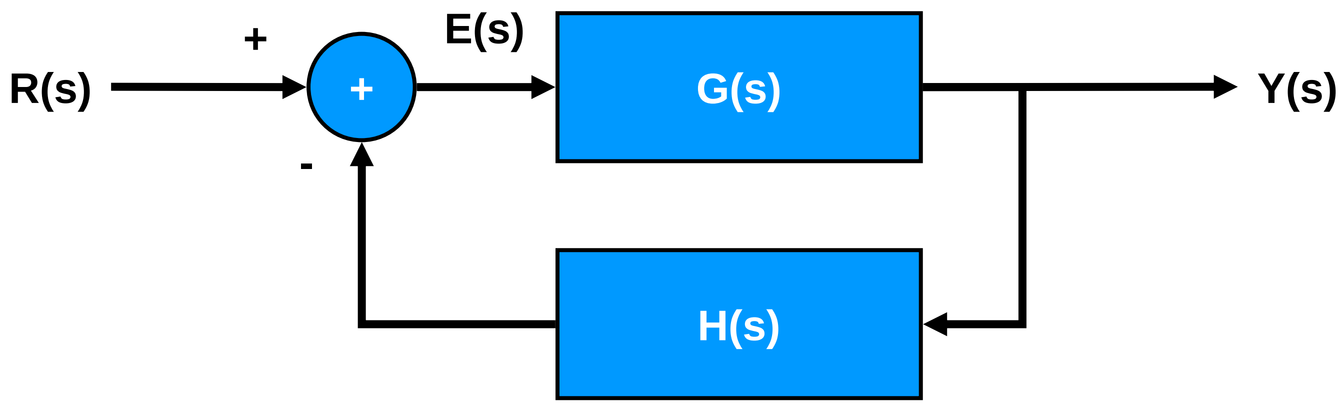
$$K_q = \lim_{s\to 0} s^2 G(s)H(s)$$

For a unit parabolic input:

$$e_{ss} = \frac{1}{K_a}$$

Type 0, Type 1, Type 2,… Systems

- With unit parabolic input $r(t)=\begin{cases}0 & t<0\\ \frac{1}{2}t^2 & t\geq 0\end{cases}$



Type 0: $$K_a=\lim_{s\to 0}\frac{s^2K(T_as+1)(T_bs+1)\ldots(T_ms+1)}{(T_1s+1)(T_2s+1)\ldots(T_ps+1)}=0 \qquad e_{ss}=\frac{1}{K_a}\to\infty$$

Type 1: $$K_a=\lim_{s\to 0}\frac{s^2K(T_as+1)(T_bs+1)\ldots(T_ms+1)}{s(T_1s+1)(T_2s+1)\ldots(T_ps+1)}=0 \qquad e_{ss}=\frac{1}{K_a}\to\infty$$

Type 2: $$K_a=\lim_{s\to 0}\frac{s^2K(T_as+1)(T_bs+1)\ldots(T_ms+1)}{s^2(T_1s+1)(T_2s+1)\ldots(T_ps+1)}=K \qquad e_{ss}=\frac{1}{K}$$

Type 3 or higher: $$K_a=\lim_{s\to 0}\frac{s^2K(T_as+1)(T_bs+1)\ldots(T_ms+1)}{s^N(T_1s+1)(T_2s+1)\ldots(T_ps+1)}\to\infty \qquad e_{ss}=\frac{1}{K_a}=0$$

Steady State Error

	Step Input r(t) = 1	Ramp input r(t)=t	Acceleration input r(t) = ½ t^2
Type 0 System	$\frac{1}{1+K}$	∞	∞
Type 1 System	0	$\frac{1}{K}$	∞
Type 2 System	0	0	$\frac{1}{K}$

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