

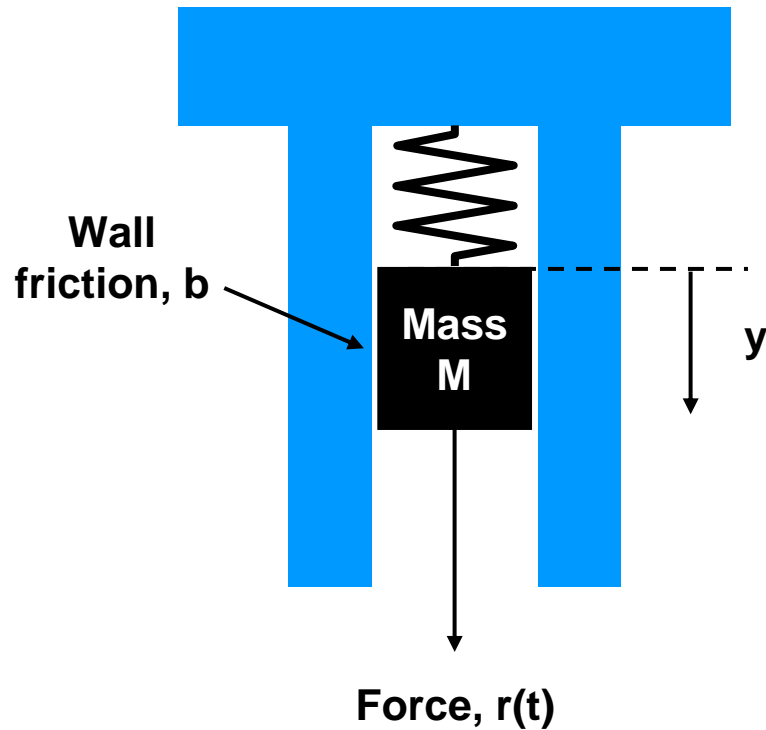
Design IV

E232 Spring 07

Class 13

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Analyzing Dynamic Systems – Differential Equations



$$M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t)$$

Using the Laplace Transform to Replace Differential Equations

- Simplify:

$$M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t)$$

$$M \left(s^2 Y(s) - sy(0-) - \frac{dy(0-)}{dt} \right) + b(sY(s) - y(0-)) + kY(s) = R(s)$$

$$r(t) = 0$$

$$y(0-) = y_0$$

$$\left. \frac{dy}{dt} \right|_{t=0-} = 0$$

$$Ms^2 Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

Using the Laplace Transform to Solve Differential Equations

- Rearrange to solve for $Y(s)$

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}$$

**Determines
“characteristic equation”**



- Example, let $k/M=5$, $b/M=6$

$$Y(s) = \frac{\left(s + \frac{b}{M}\right)y_0}{s^2 + \frac{b}{M}s + \frac{k}{M}} = \frac{(s + 6)y_0}{(s + 5)(s + 1)} = \frac{p(s)}{q(s)}$$

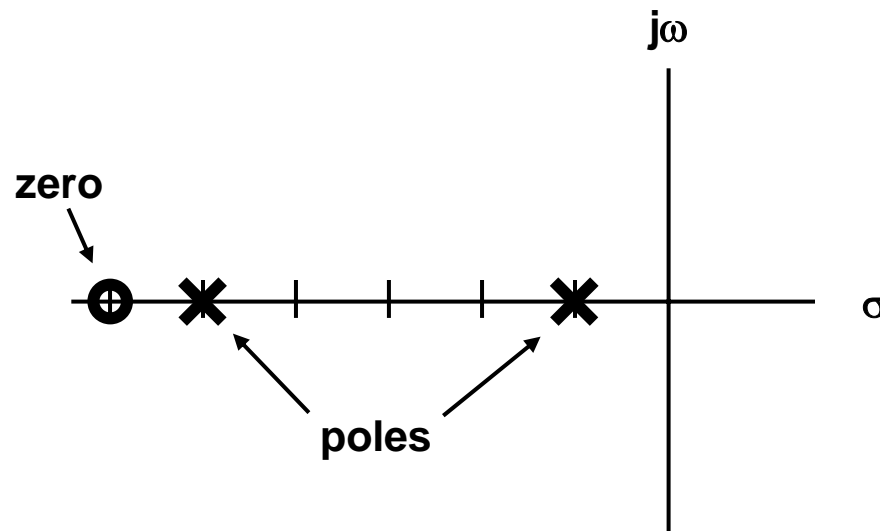
Using the Laplace Transform to Solve Differential Equations

- “Poles” and “zeroes” of $Y(s)$

$$Y(s) = \frac{(s + 6)y_0}{(s + 5)(s + 1)} = \frac{p(s)}{q(s)}$$

At $s=-6$, $Y(s)=0$

At $s=-1$ or $s=-5$, $Y(s)$ increases without bound



Using the Laplace Transform to Solve Differential Equations

- Partial fraction expansion of $Y(s)$, assume $y_0=1$

$$Y(s) = \frac{(s+6)y_0}{(s+5)(s+1)} = \frac{k_1}{s+1} + \frac{k_2}{s+5}$$

$$k_1 = \frac{(s-s_1)p(s)}{q(s)} \Big|_{s=s_1} = \frac{(s+1)(s+6)}{(s+5)(s+1)} \Big|_{s=-1} = \frac{5}{4}$$

$$k_2 = \frac{(s-s_2)p(s)}{q(s)} \Big|_{s=s_2} = \frac{(s+5)(s+6)}{(s+5)(s+1)} \Big|_{s=-5} = \frac{1}{-4} = -\frac{1}{4}$$

Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$Y(s) = \frac{(s+6)}{(s+5)(s+1)} = \frac{\frac{5}{4}}{s+1} + \frac{-\frac{1}{4}}{s+5}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{5}{4}}{s+1} + \frac{-\frac{1}{4}}{s+5}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\frac{5}{4}}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s+5}\right\}$$

Using the Laplace Transform to Solve Differential Equations

- Find inverse Laplace Transform

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{5/4}{s+1} \right\} + \mathcal{L}^{-1} \left\{ \frac{-1/4}{s+5} \right\}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$y(t) = \frac{5}{4} e^{-t} - \frac{1}{4} e^{-5t}$$

Using the Laplace Transform to Solve Differential Equations

- Final value theorem:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$

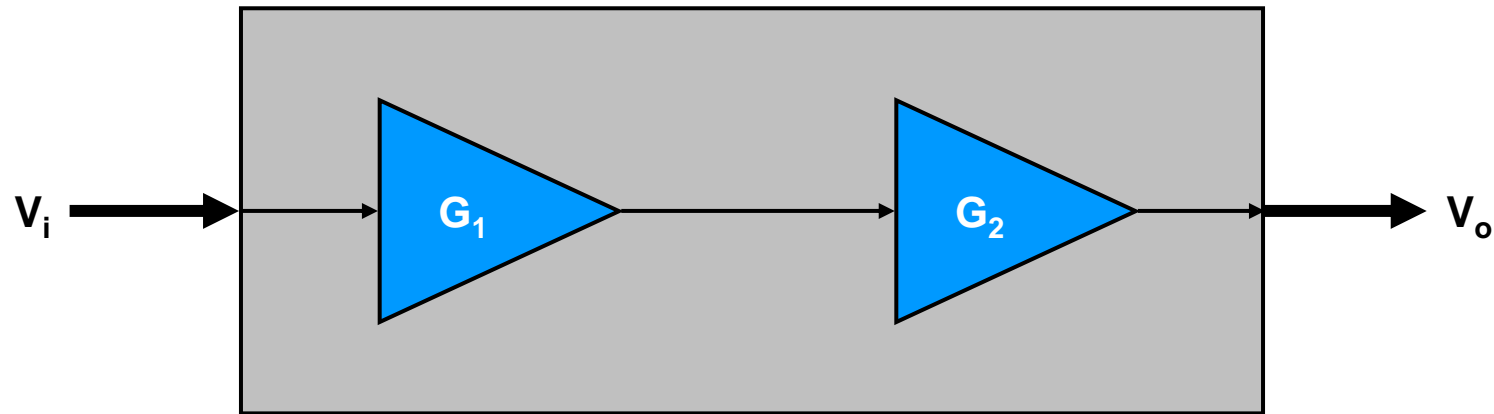
$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{s(s+6)}{(s+5)(s+1)} = \frac{0 \cdot 6}{5 \cdot 1} = 0$$

Today's topics

- Control systems
 - Transfer function
 - Steady state response of proportional control system

Representing Stages As Gain Blocks

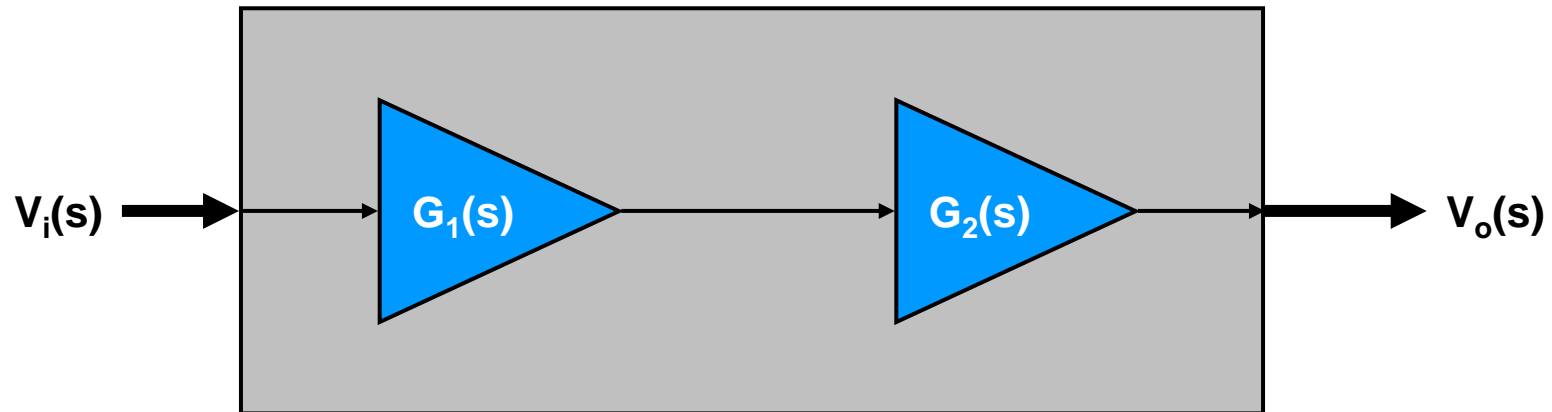
- Consider two amplifiers in cascade



$$G_{total} = \frac{V_o}{V_i} = G_1 G_2$$

Representing Stages As Transfer Function Blocks

- Generalize G to be a function of s

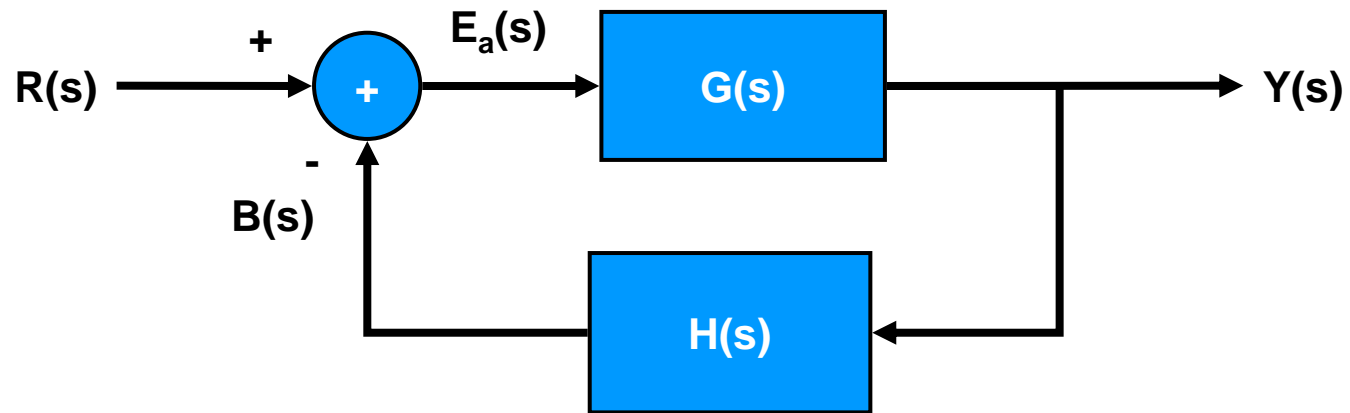


$$G_{total}(s) = \frac{V_o(s)}{V_i(s)} = G_1(s)G_2(s)$$

As long as the functions are linear

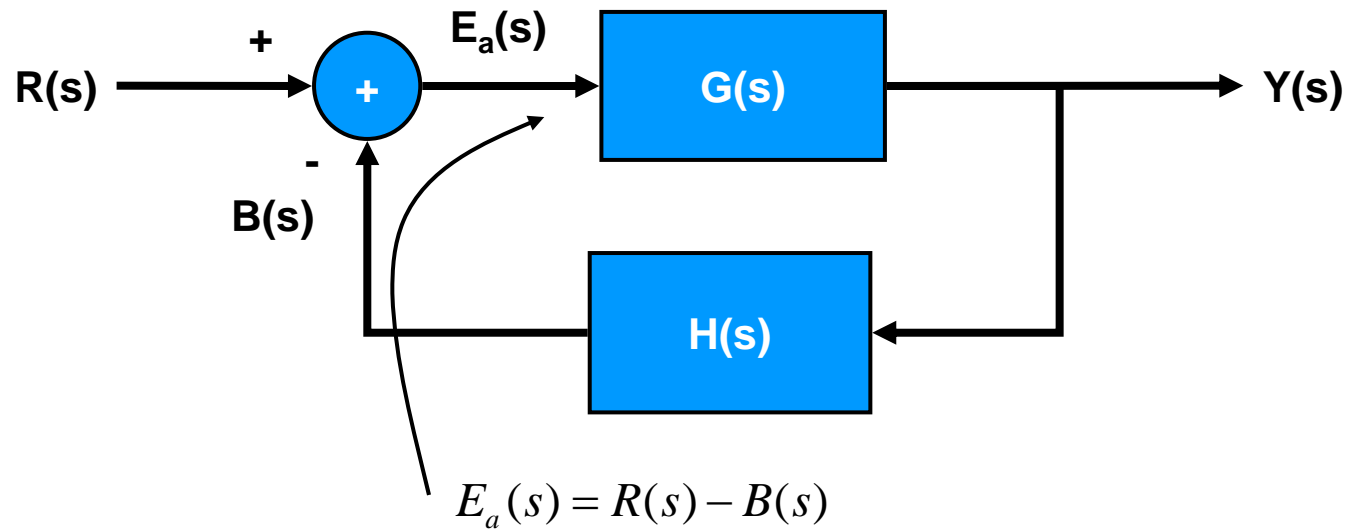
Transfer Function of a Feedback System

- Consider a generic feedback control system



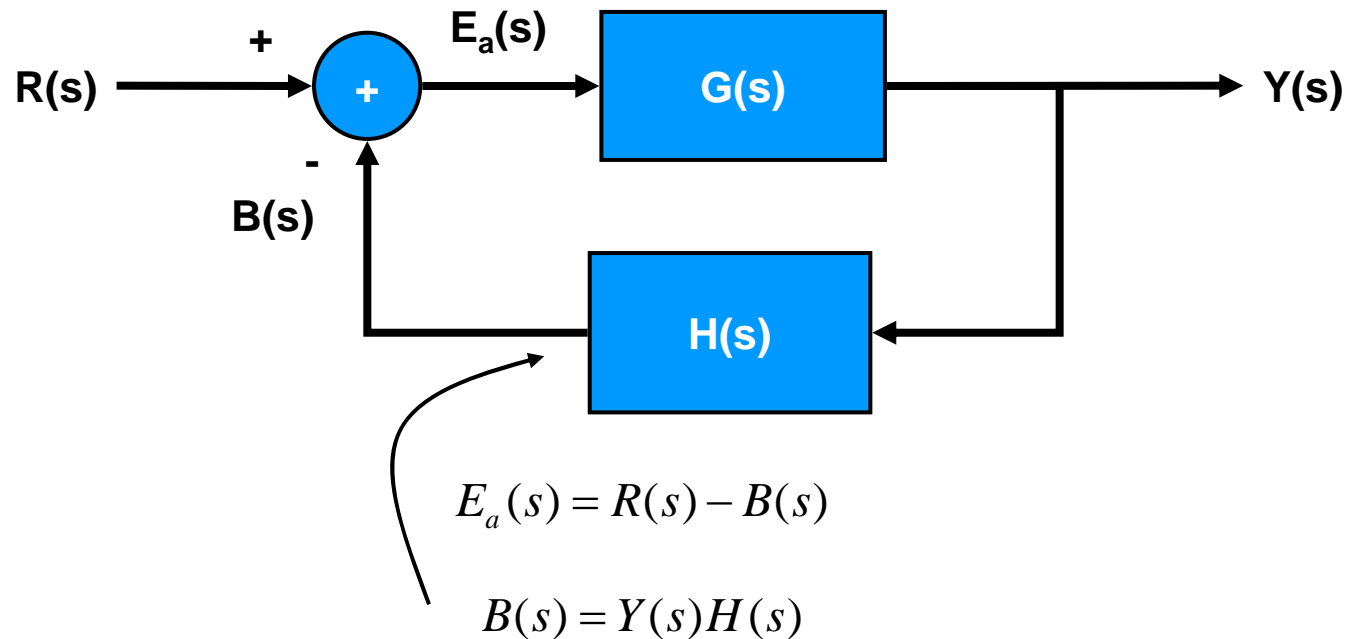
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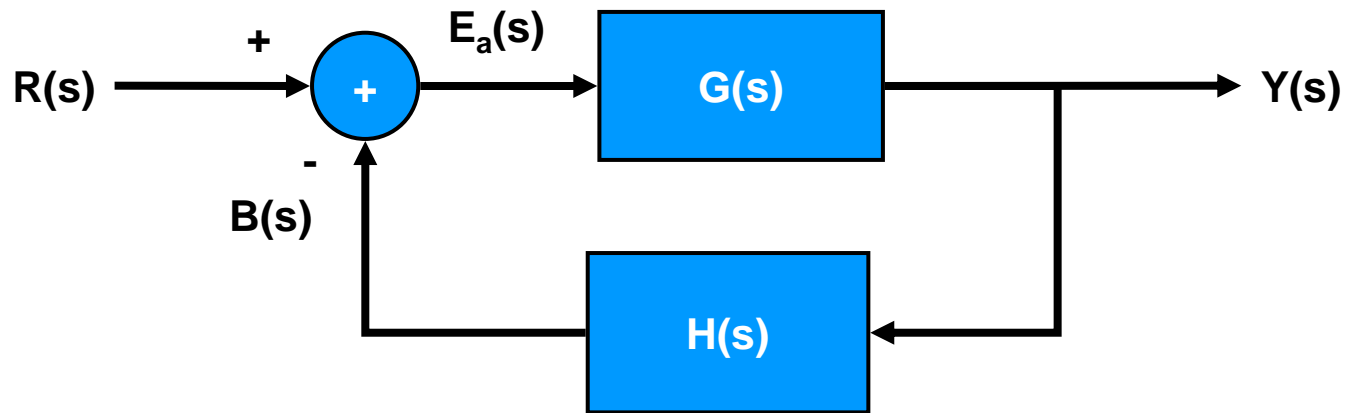
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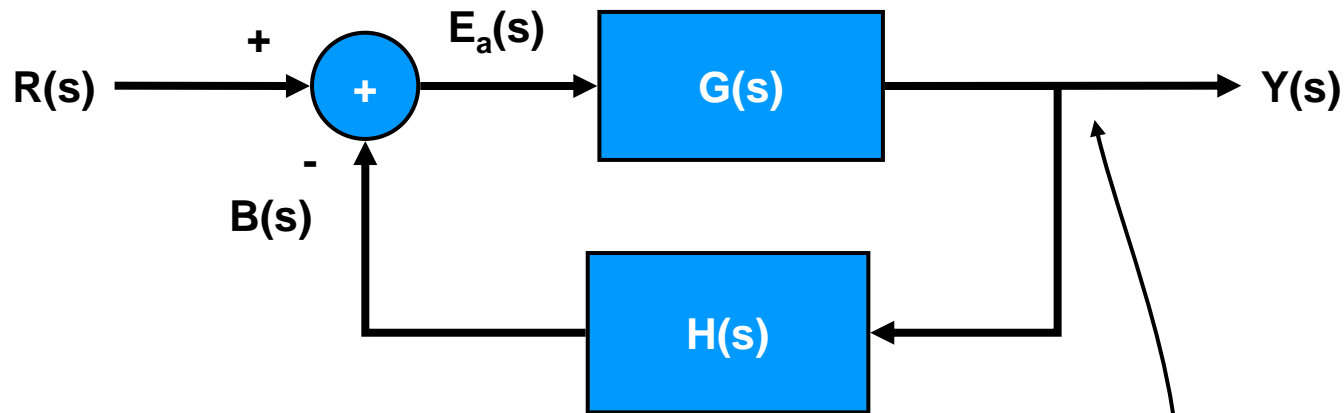
$$E_a(s) = R(s) - B(s)$$

$$B(s) = Y(s)H(s)$$

$$E_a(s) = R(s) - Y(s)H(s)$$

Transfer Function of a Feedback System

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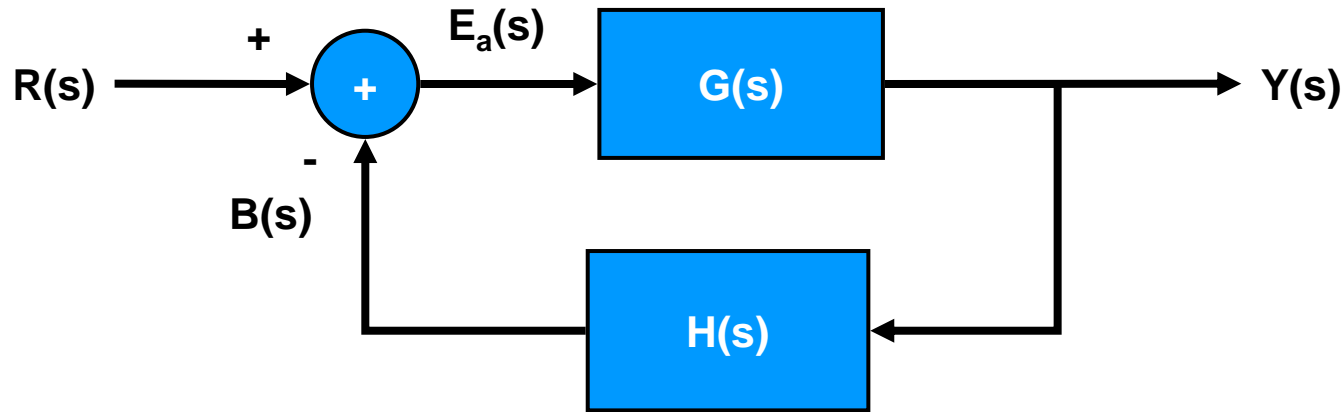
$$B(s) = Y(s)H(s)$$

$$E_a(s) = R(s) - Y(s)H(s)$$

$$Y(s) = E_a(s)G(s)$$

Transfer Function of a Feedback System

- Consider a generic feedback control system



$$E_a(s) = R(s) - B(s) \quad Y(s) = (R(s) - Y(s)H(s))G(s)$$

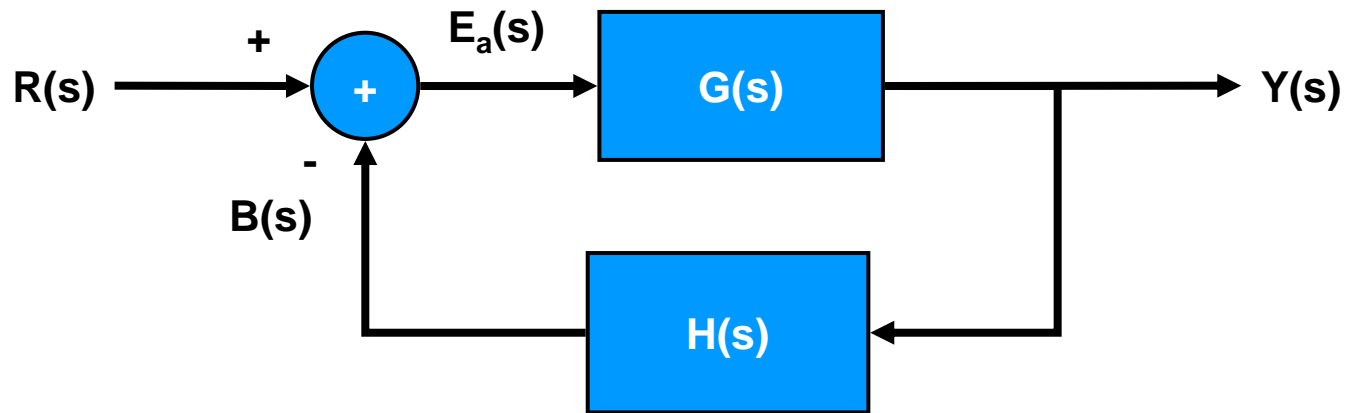
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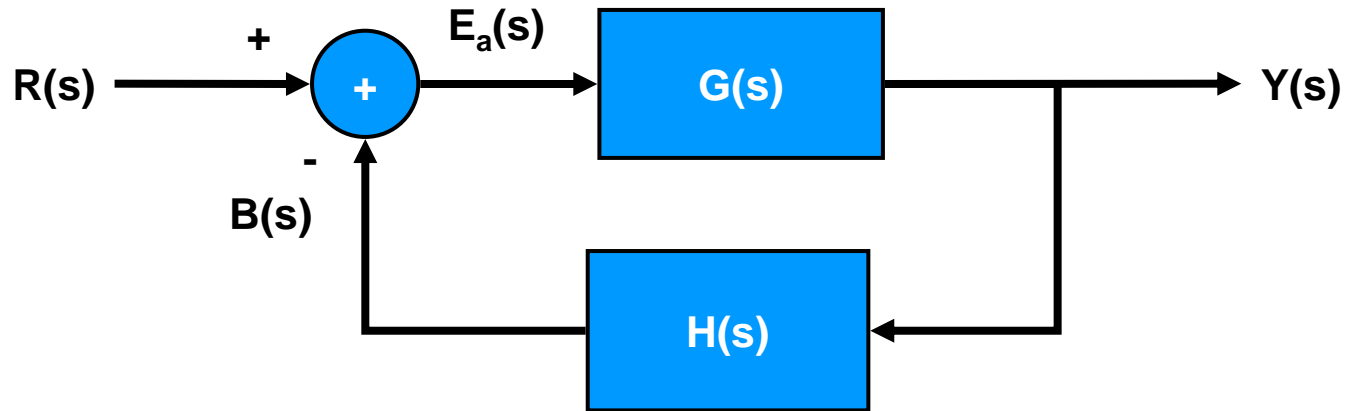


$$Y(s) = (R(s) - Y(s)H(s))G(s)$$

$$Y(s)[1 + G(s)H(s)] = R(s)G(s)$$

Transfer Function of a Feedback System

- Consider a generic feedback control system



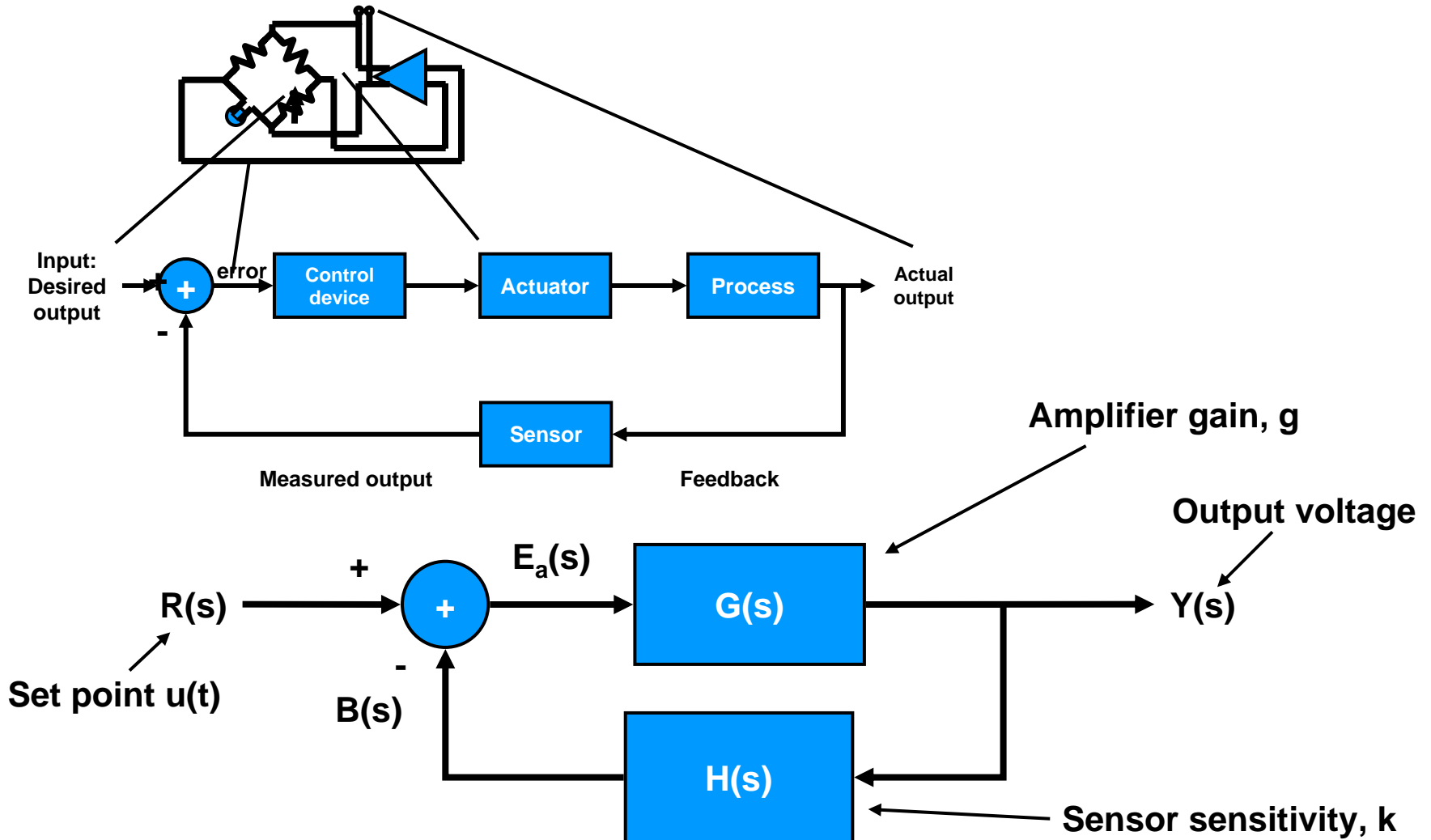
$$Y(s) = (R(s) - Y(s)H(s))G(s)$$

$$Y(s)[1 + G(s)H(s)] = R(s)G(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

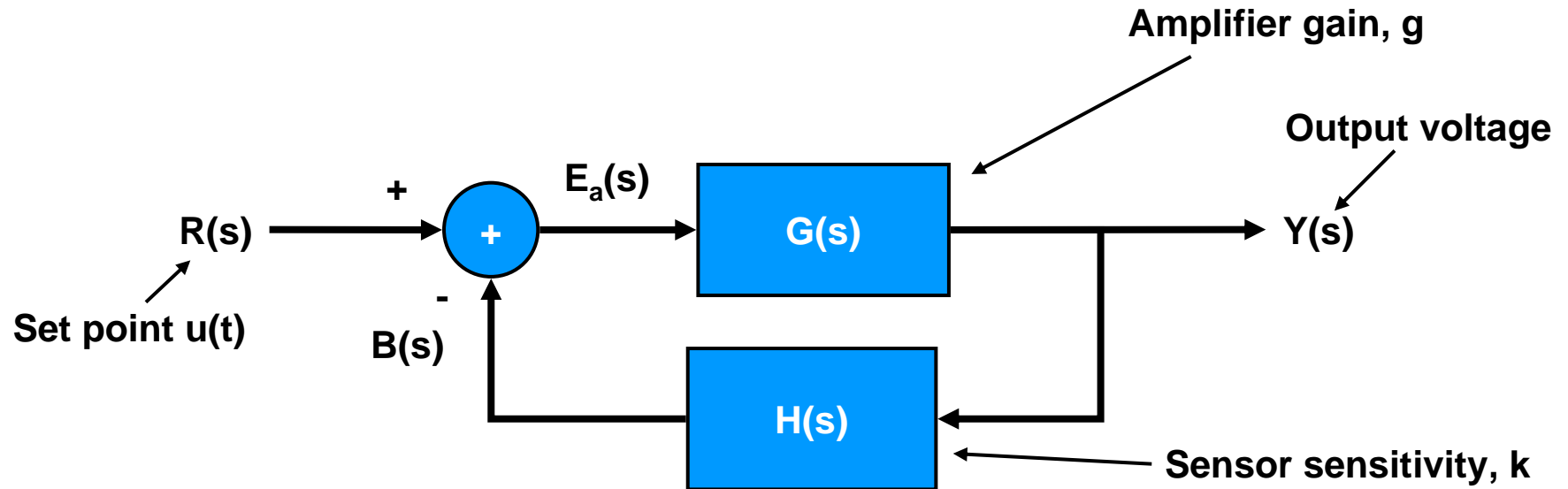
Steady State Response

- Consider the hot-wire anemometer as a feedback control system



Steady State Response

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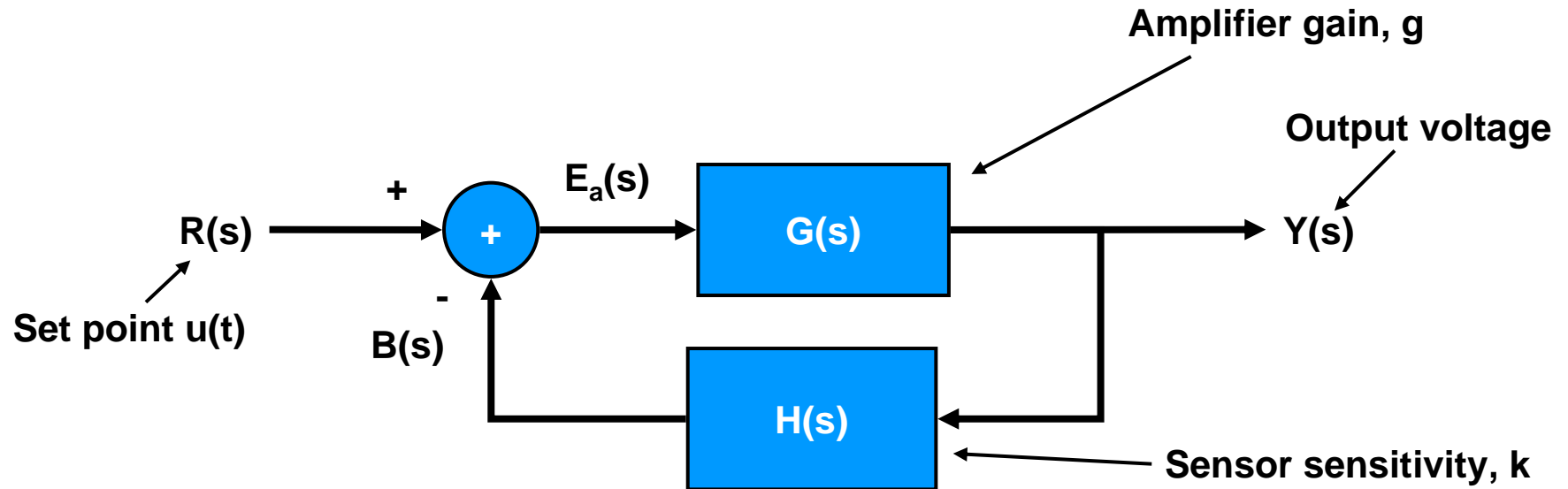


$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system



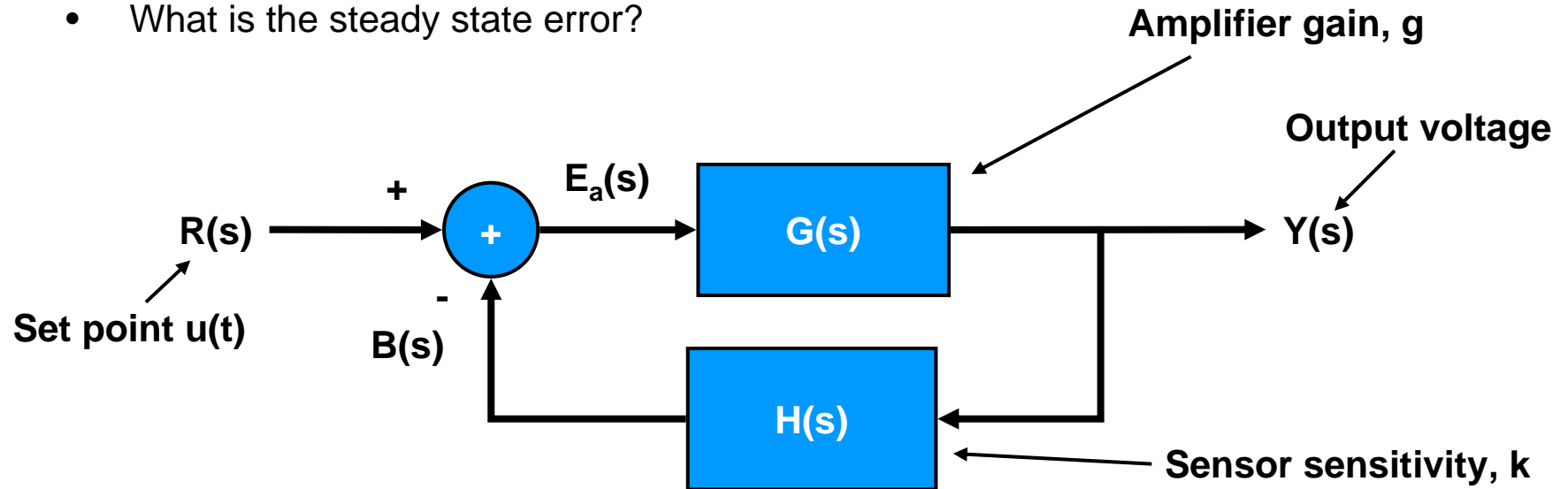
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$Y(s) = \frac{g}{1 + gk} \frac{1}{s}$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state error?



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

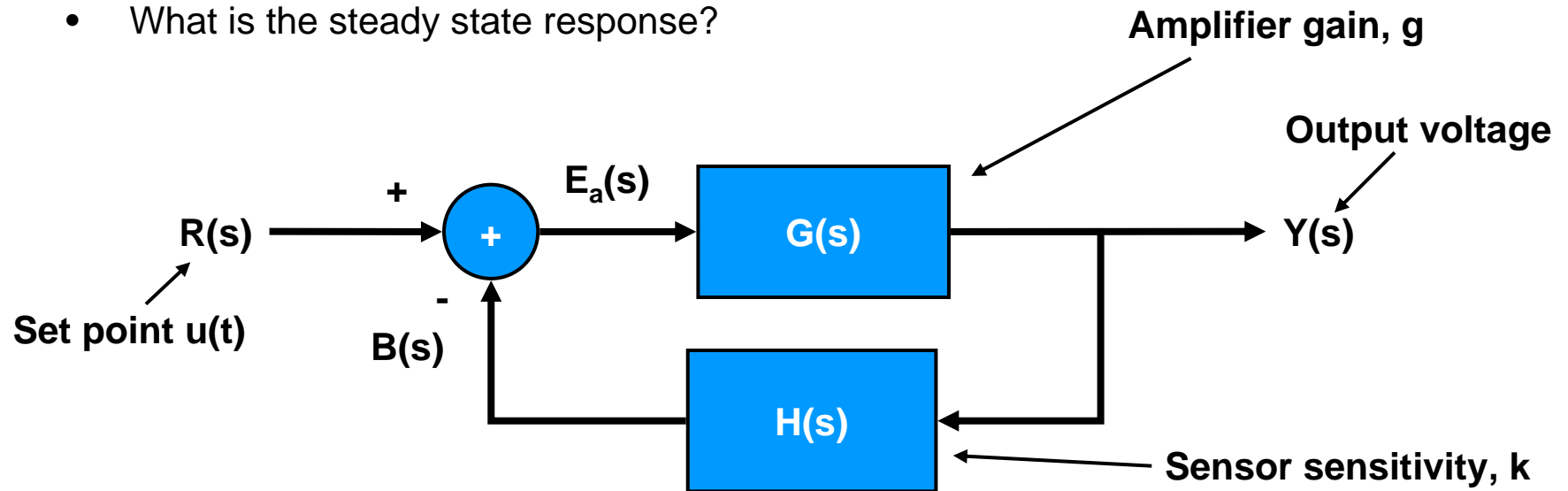
$$Y(s) = \frac{g}{1 + gk} \frac{1}{s}$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

$$E_a(s) = \frac{g}{1 + gk} \frac{1}{s} \frac{1}{g}$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?



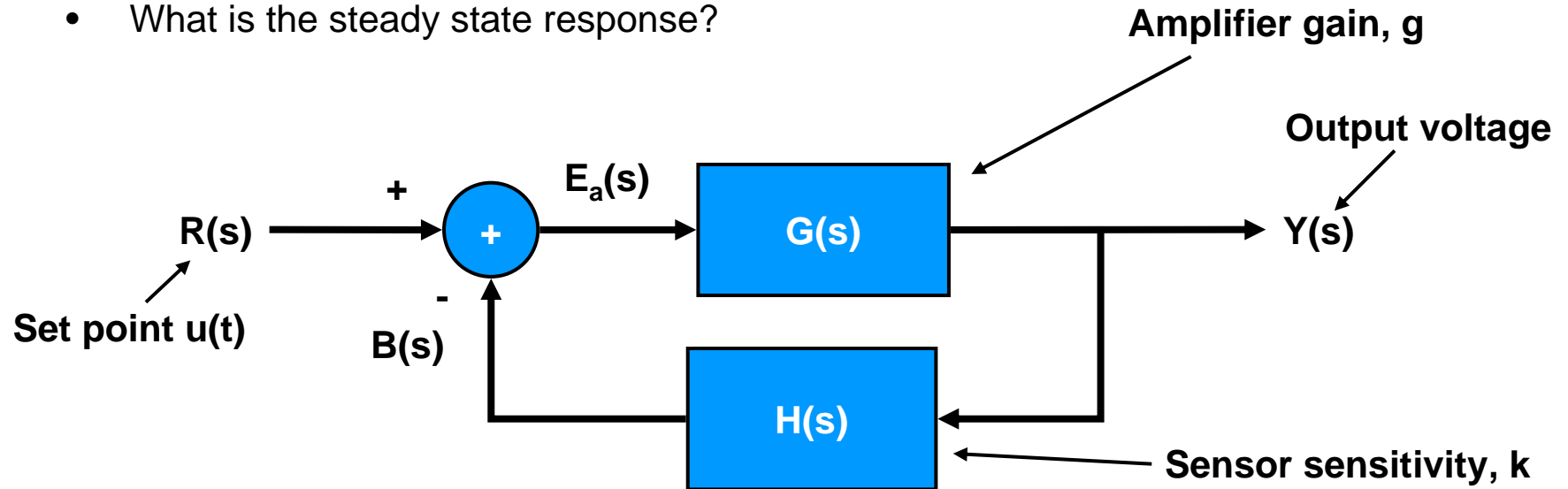
$$E_a(s) = \frac{1}{s(1+gk)}$$

From the final value theorem:

$$\lim_{t \rightarrow \infty} z(t) = \lim_{s \rightarrow 0} sZ(s)$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
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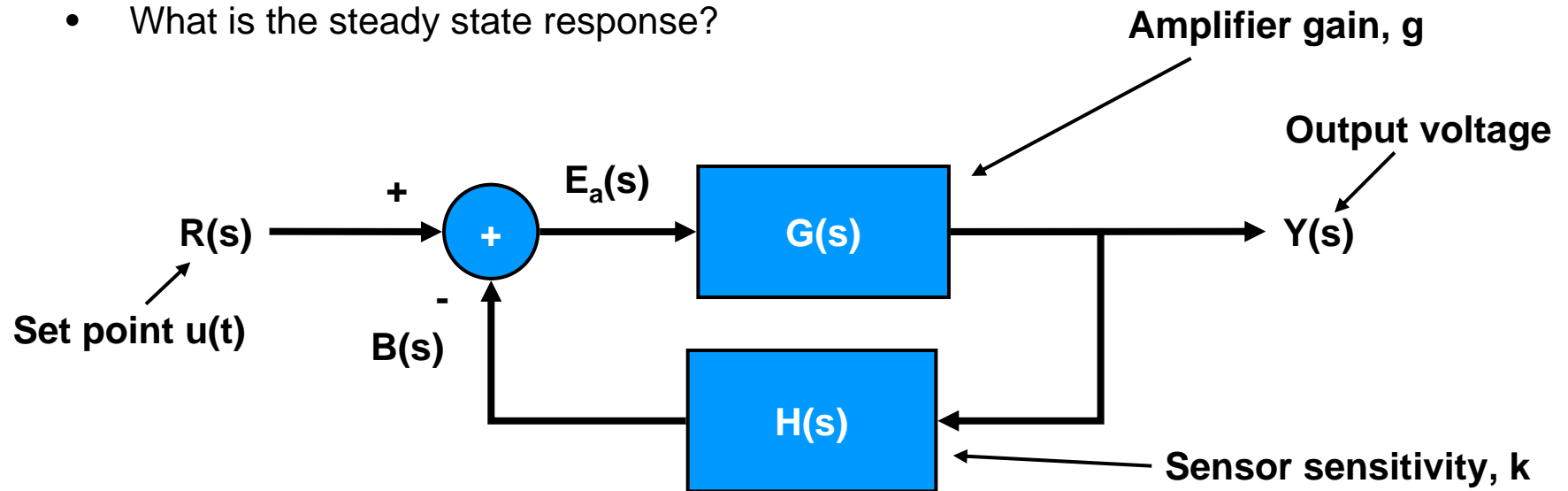
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Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?



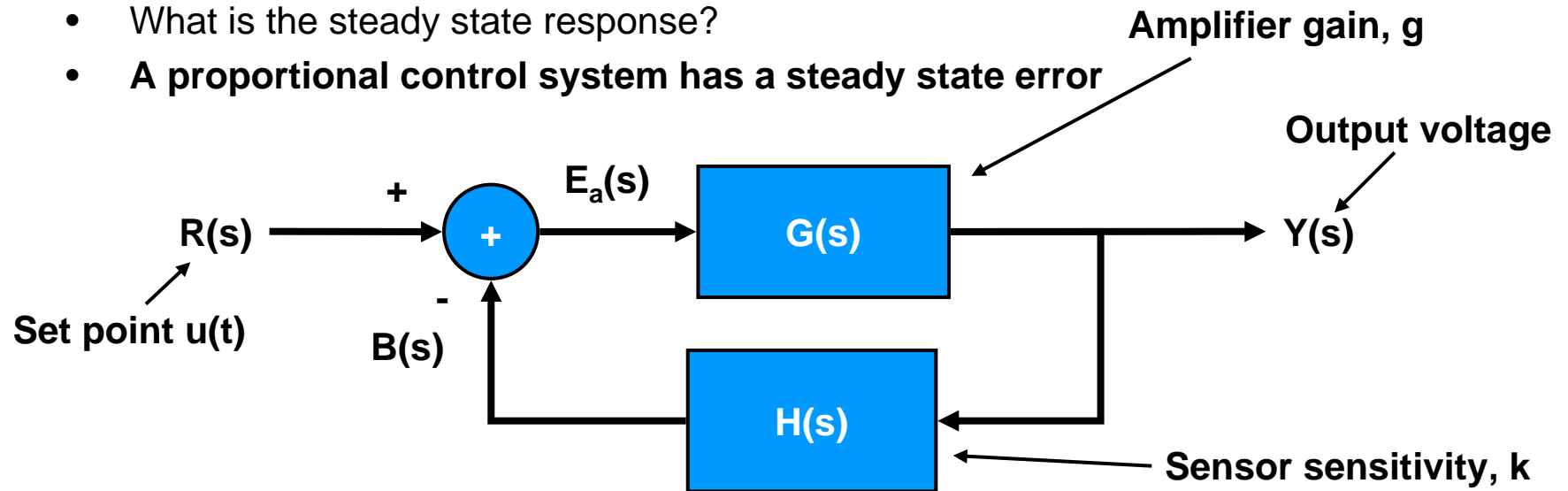
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$$\lim_{t \rightarrow \infty} z(t) = \lim_{s \rightarrow 0} sZ(s) \quad \lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} s \frac{1}{s(1 + gk)} \quad \lim_{t \rightarrow \infty} e_a(t) = \frac{1}{1 + gk}$$

Steady State Response

- Consider the hot-wire anemometer as a feedback control system
- What is the steady state response?
- **A proportional control system has a steady state error**



$$Y(s) = \frac{g}{1 + gk} \frac{1}{s}$$

From the final value theorem:

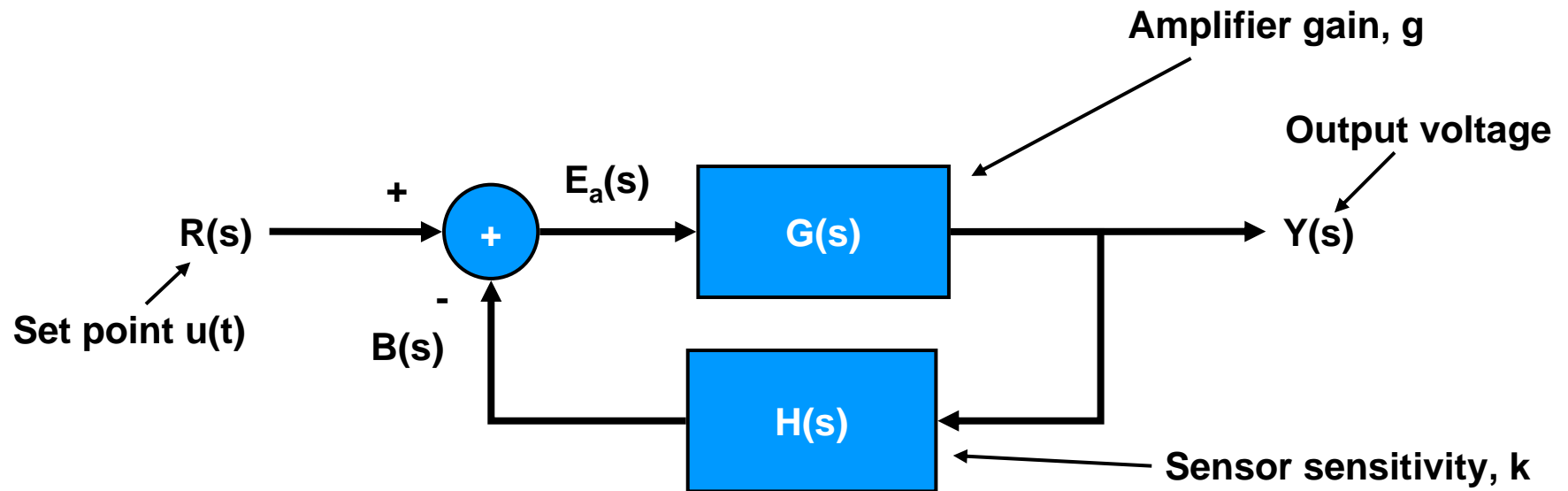
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$$\lim_{t \rightarrow \infty} e_a(t) = \frac{1}{1 + gk}$$

Steady State Response

- To reduce the steady state error:

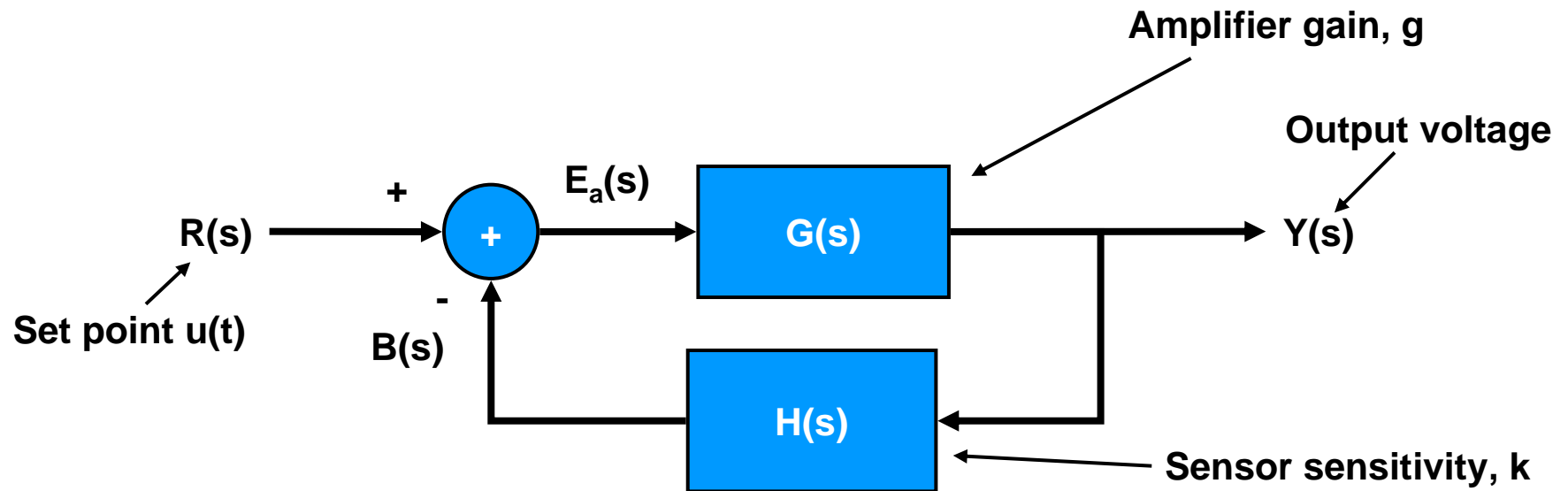


$$\lim_{t \rightarrow \infty} e_a(t) = \frac{1}{1 + gk}$$

Increase gain, either in amplifier or sensor

Steady State Response

- To reduce the steady state error:

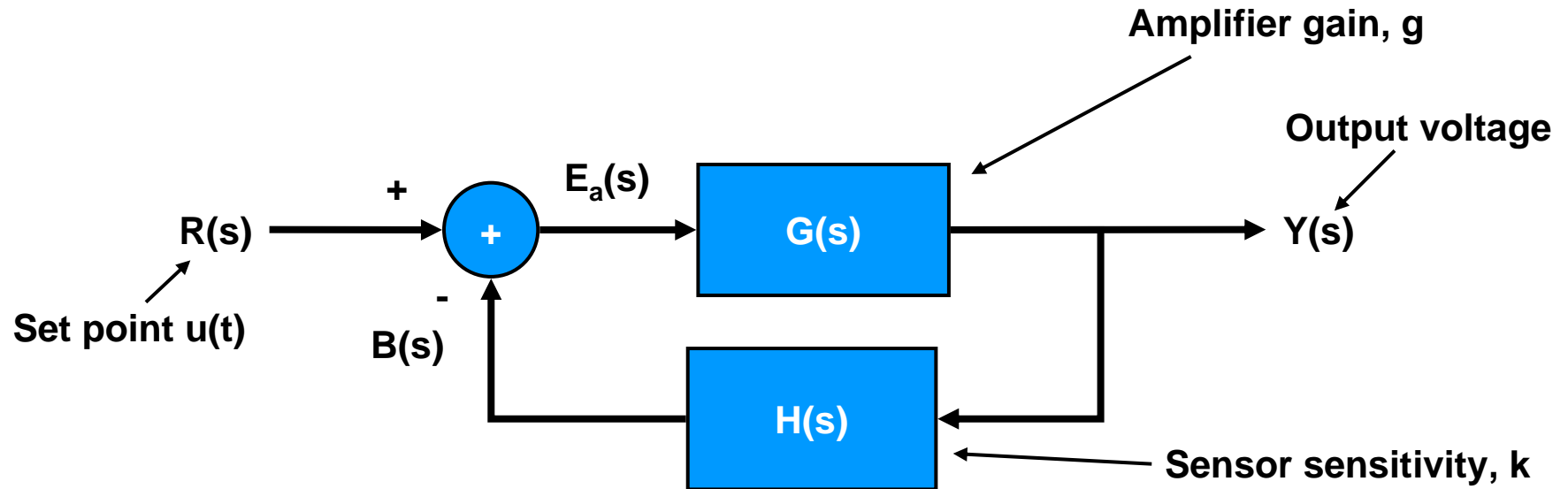


$$\lim_{t \rightarrow \infty} e_a(t) = \frac{1}{1 + gk}$$

Increase gain, either in amplifier or sensor
Making system more susceptible to noise!

Steady State Response

- To reduce the steady state error:

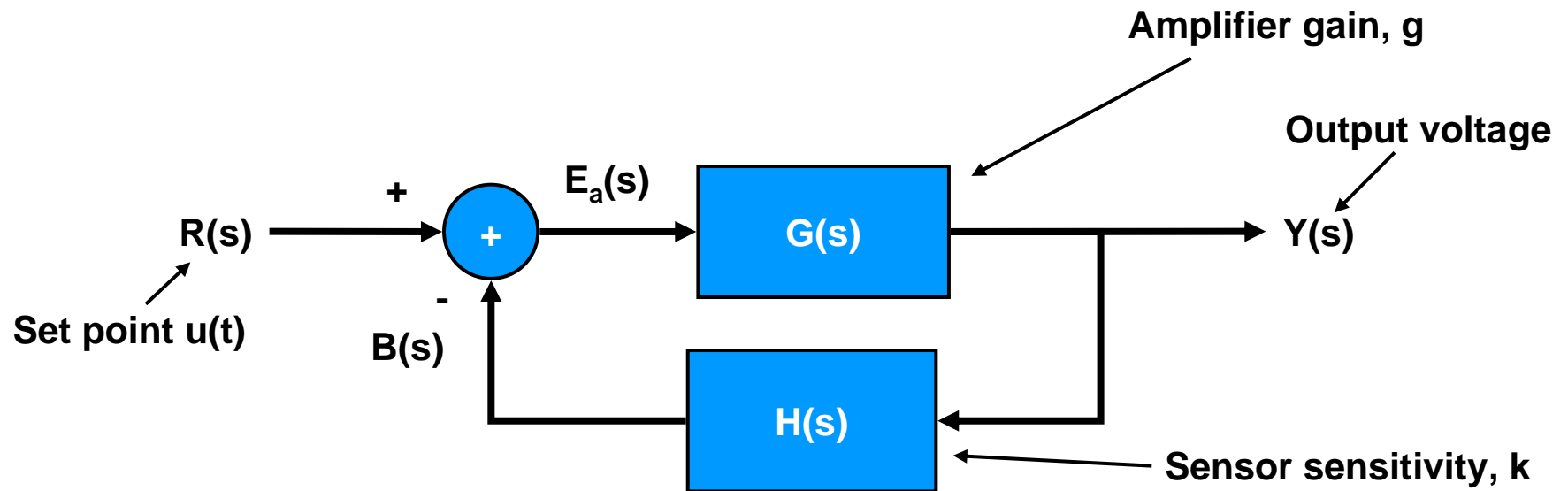


But, what if error signal was something like:

$$\lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} s \frac{1}{s \left(1 + g \frac{k}{s} \right)}$$

Steady State Response

- To reduce the steady state error:



But, what if error signal was something like:

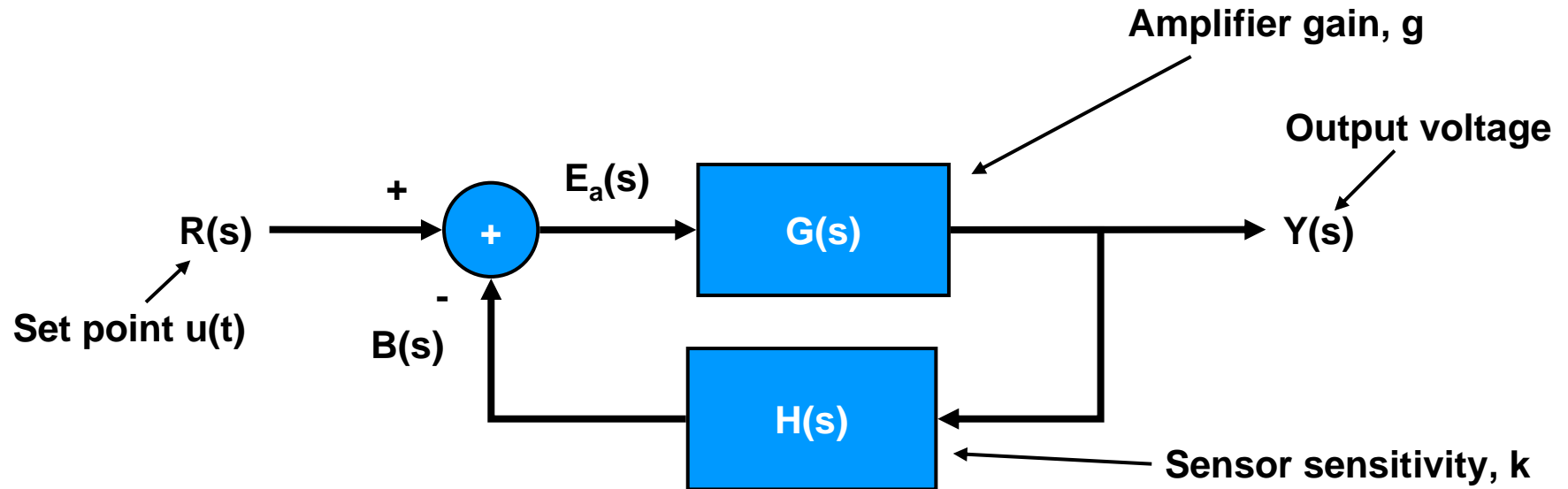
$$\lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} s \frac{1}{s \left(1 + g \frac{k}{s} \right)}$$

Then:

$$\lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} \frac{1}{\left(1 + g \frac{k}{s} \right)} = \frac{s}{s + gk} = 0$$

Steady State Response

- To reduce the steady state error:



But, what if error signal was something like:

$$\lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} s \frac{1}{s \left(1 + g \frac{k}{s} \right)}$$

Remember

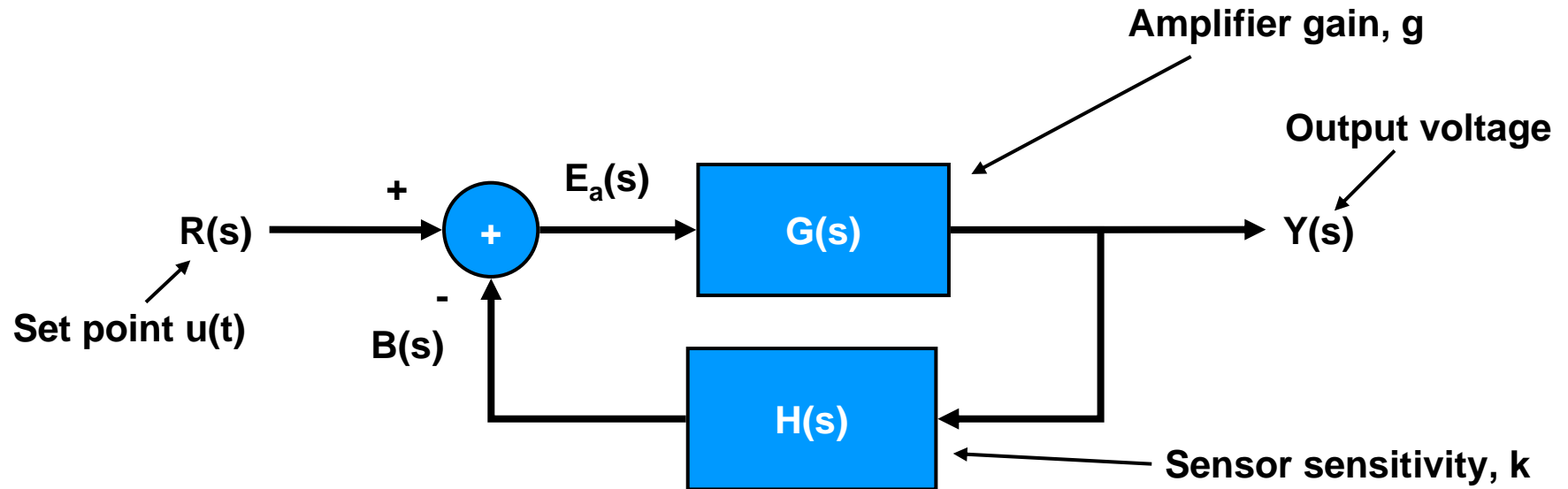
$$\frac{1}{s} \equiv \int dt$$

Then:

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Steady State Response

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Then:

$$\lim_{t \rightarrow \infty} e_a(t) = \lim_{s \rightarrow 0} \frac{1}{\left(1 + g \frac{k}{s} \right)} = \frac{s}{s + gk} = 0$$

Remember

$$\frac{1}{s} \equiv \int dt$$

Integration of error signal allows steady state error to go to zero

Next time

- Proportional, Integral and Derivative control systems

Homework 6

- Dorf & Bishop problem DP1.2, E2.18, P2.37
- Problem numbers are for reference only. The problem definition is on WebCT for copyright reasons.