# Design IV

# E232 Fall 07

Class 10

Bruce McNair
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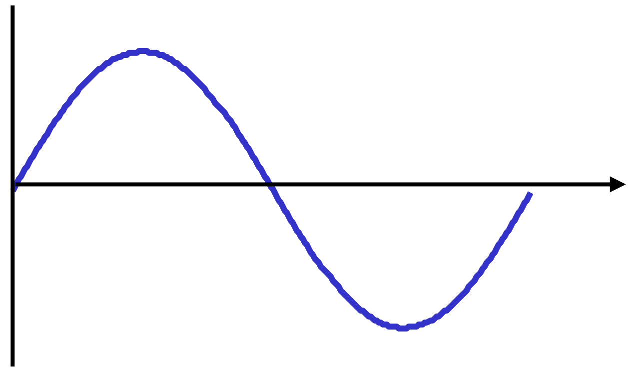
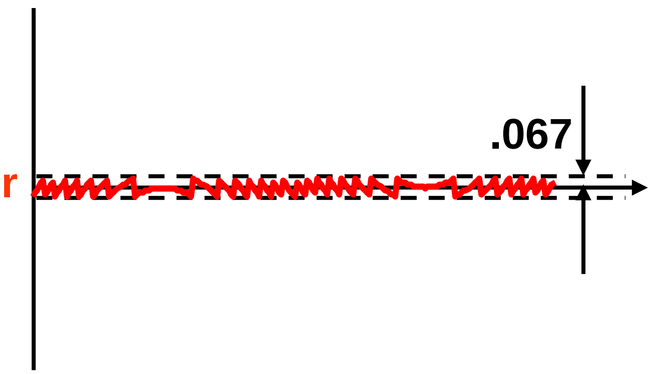
# Computerized Data Acquisition Systems

- Quantization effects

**Original analog signal**

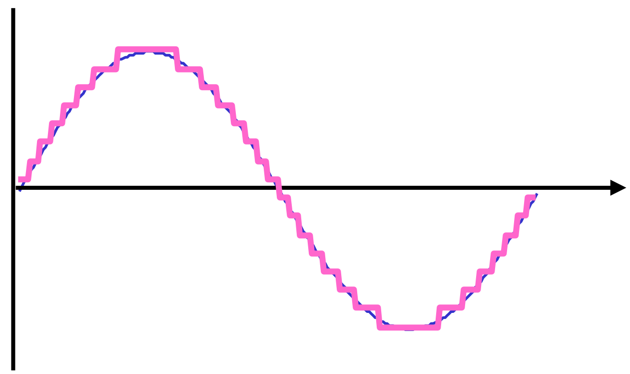


**Error**

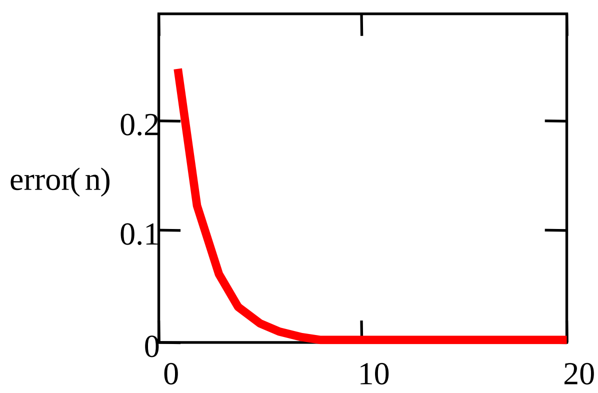


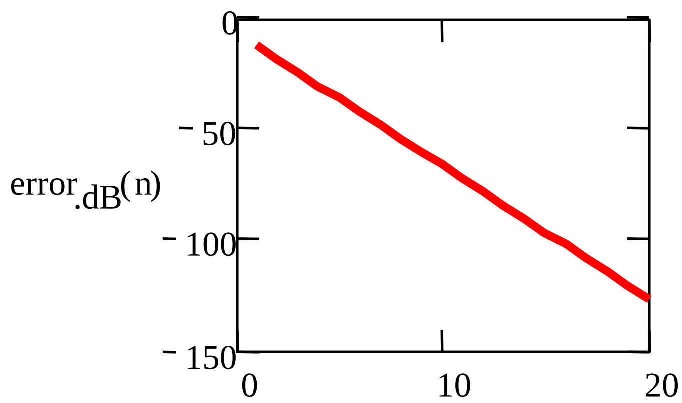

**rms(Error) = .037**

**Digital representation with 3 bits**



# Quantization Error

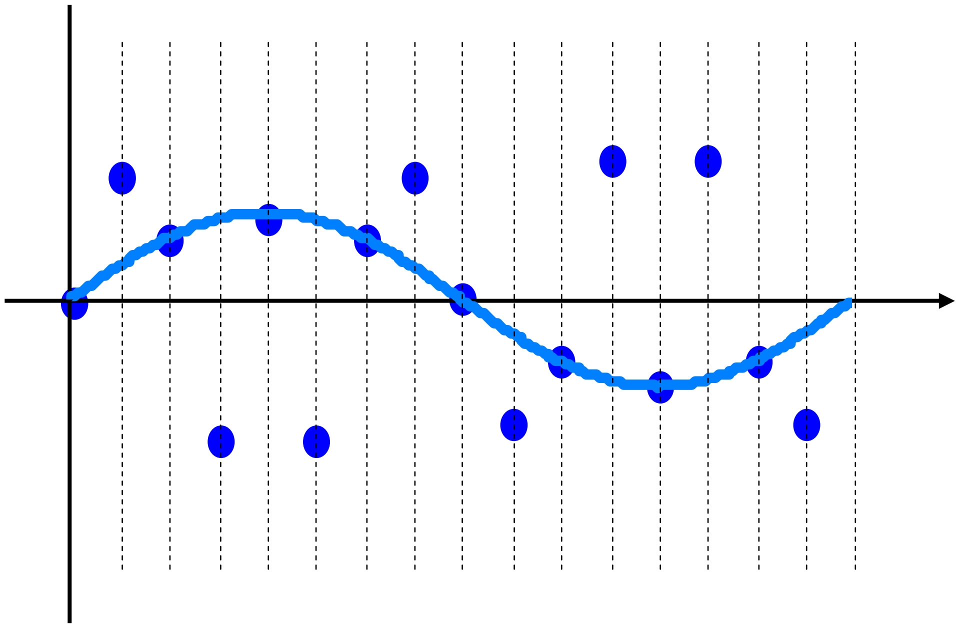



$$Q.E.(n) = \frac{.5}{2^n}$$

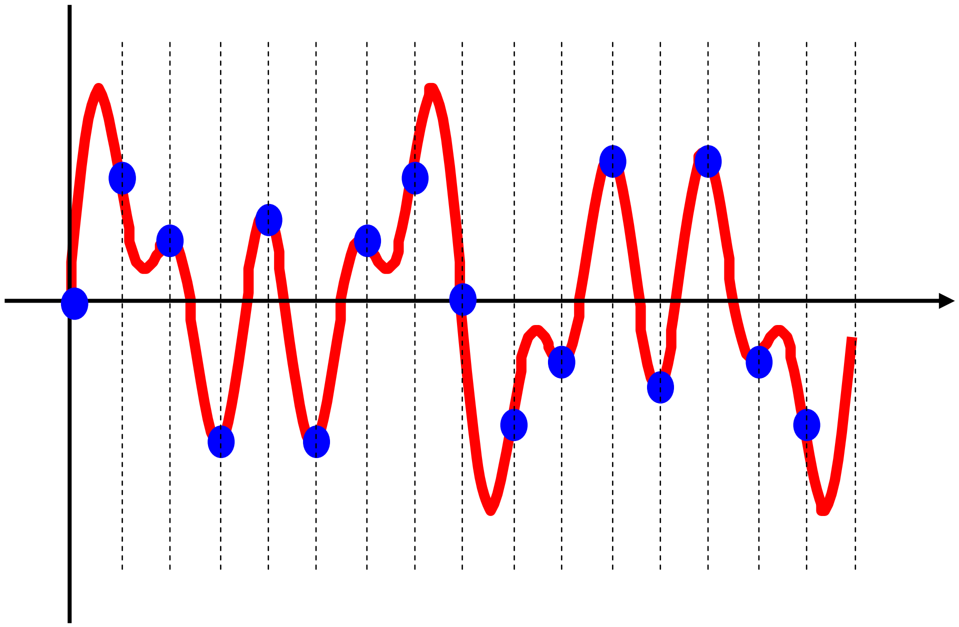
$$Q.E._{dB}(n) \approx -n \cdot 6$$

# Sampling Time-varying Signals



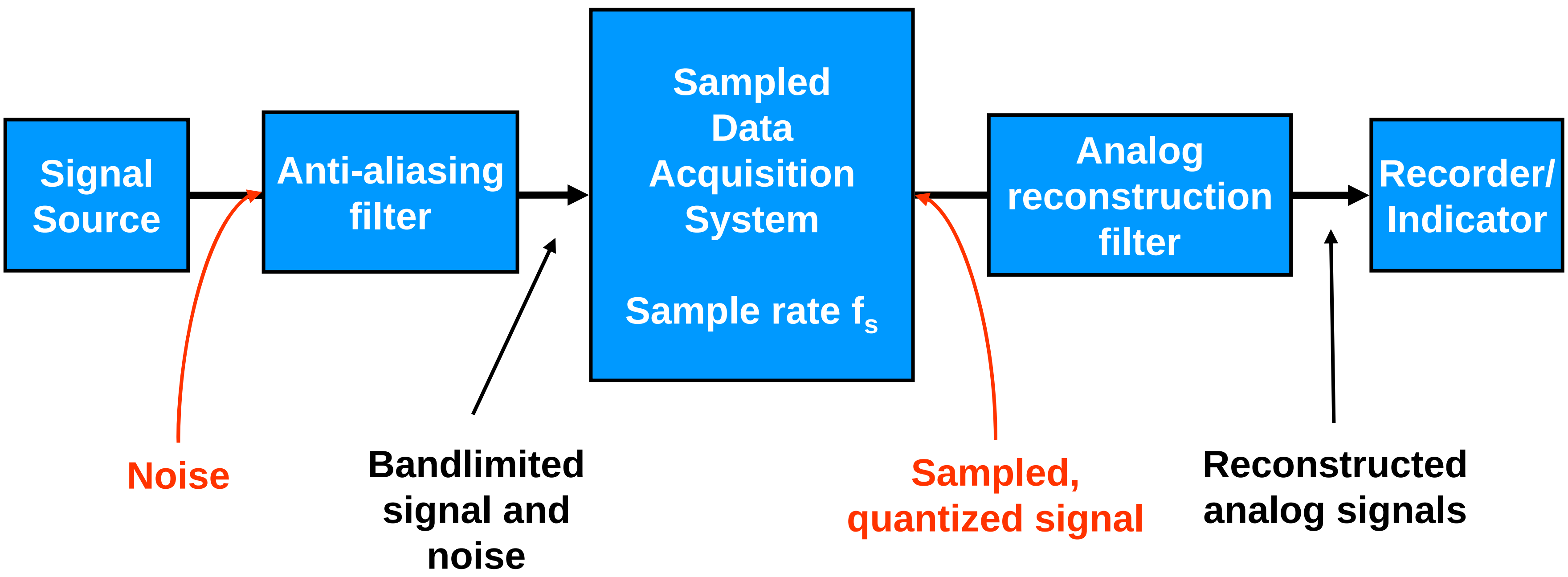
**What if we provide more samples?**

# Sampling Time-varying Signals



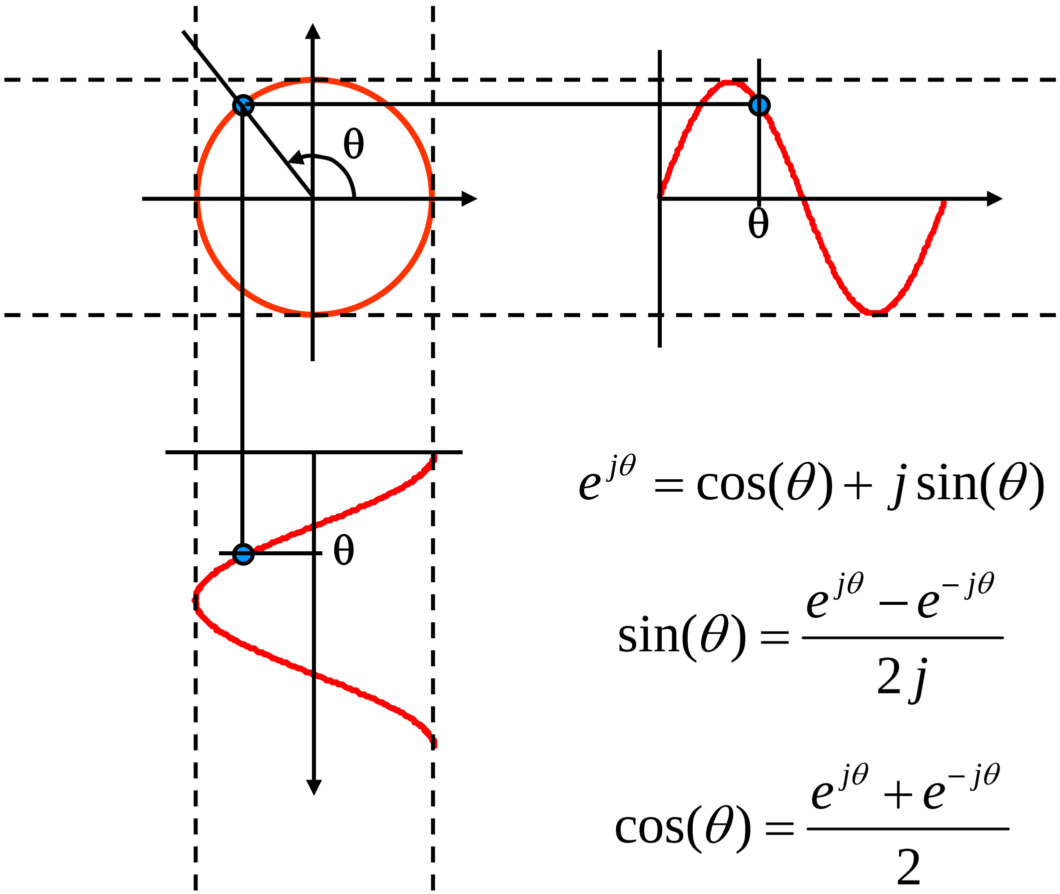
**This is what the actual signal looked like:**
**The first set of samples was at too low a frequency**

# Practical Sampling Considerations



# Frequency Domain Analysis

- Consider the functions sin(2πt), cos(2πt):




$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

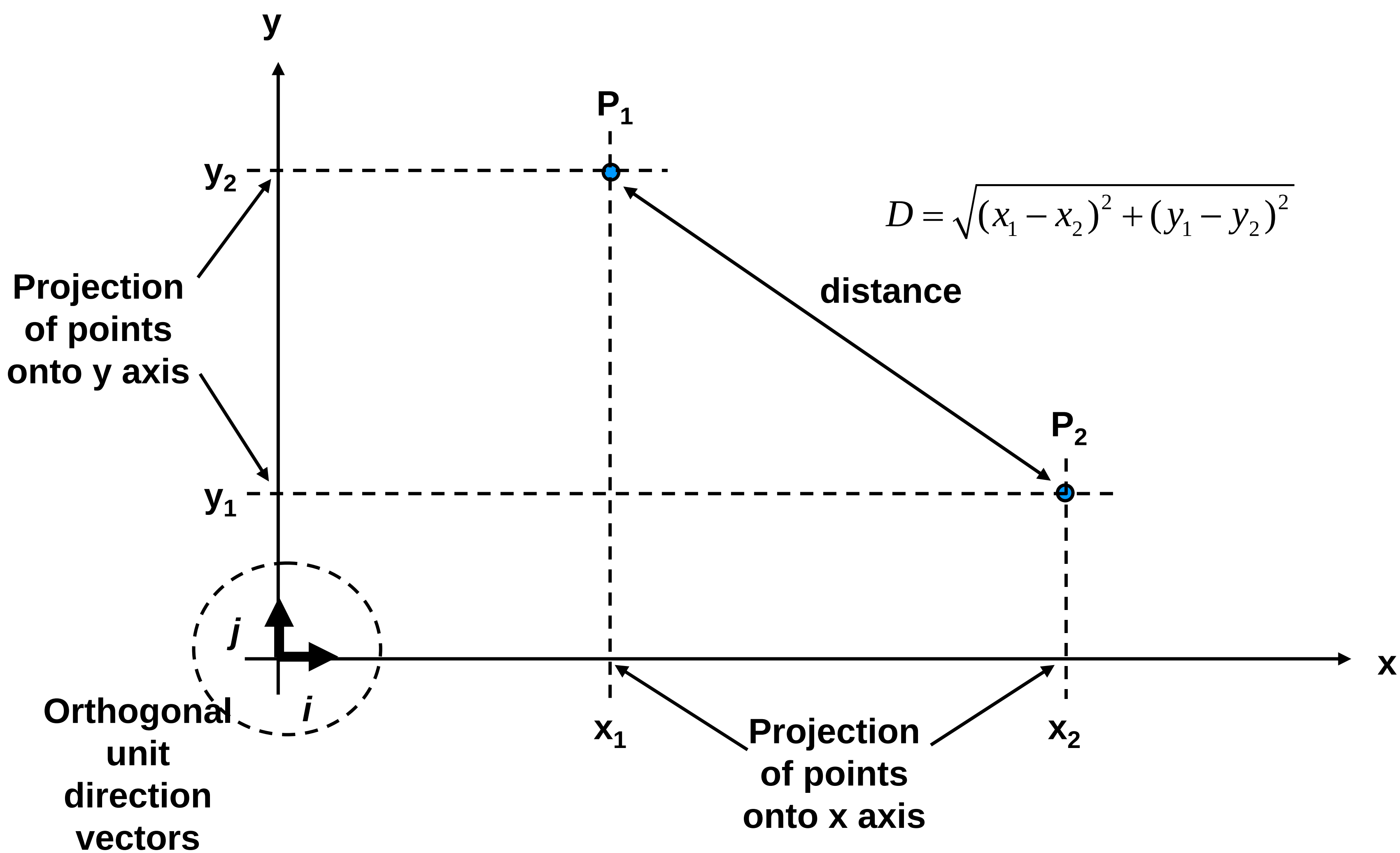
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\pi} + 1 = 0$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

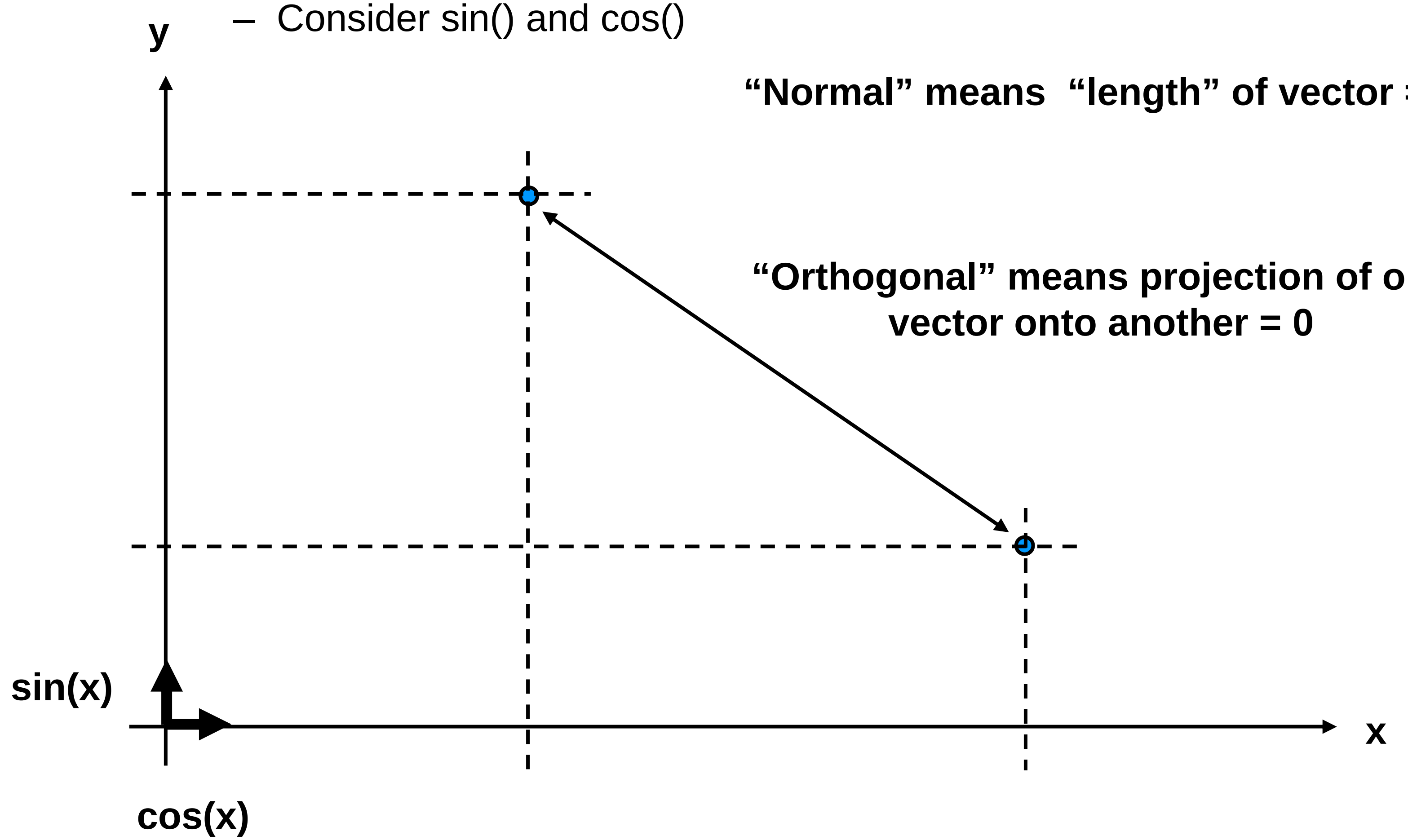
# An Aside: Measuring Distance

- Rectangular coordinate system



# An Aside: Measuring Distance

- What other ***ortho-normal*** basis vectors could be used?
  - Consider sin() and cos()



# Measuring Similarity and Distance

- The projection of A onto B, where A&B are periodic functions with period T:

$$\langle A, B \rangle = \frac{1}{T} \int_{t}^{t+T} A(x)B(x)dx$$

# Other Sinusoidal Basis Functions

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0$$

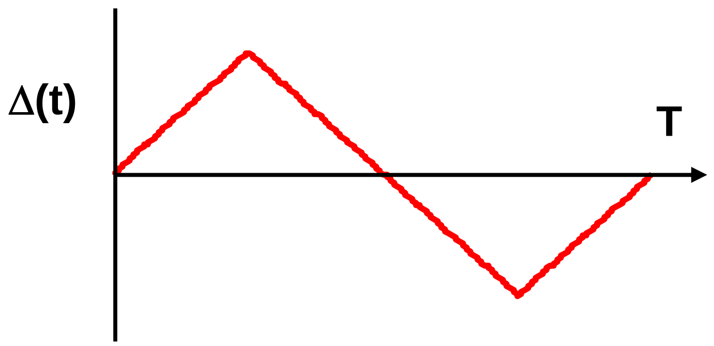
**For all m, n**

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(nx)\sin(mx)dx = 0$$

**Unless m=n**

# Spectral Analysis of Arbitrary Signals

- Consider a triangular wave with a period T:



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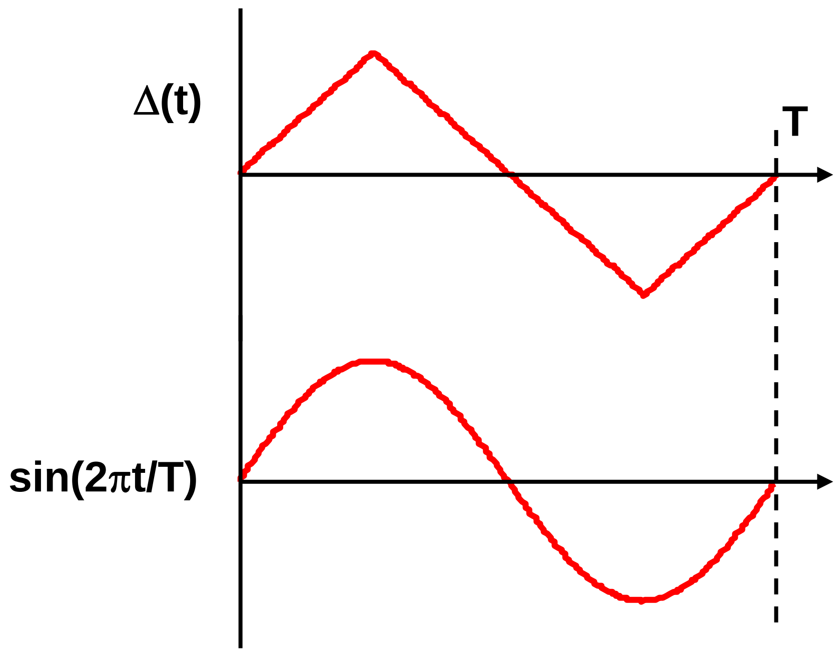


**How similar is this signal to a sinusoid with the same period?**

$$\frac{1}{\pi}\int_0^T \Delta(t)\sin\left(\frac{2\pi t}{T}\right)dt = 0.811$$

# Spectral Analysis of Arbitrary Signals

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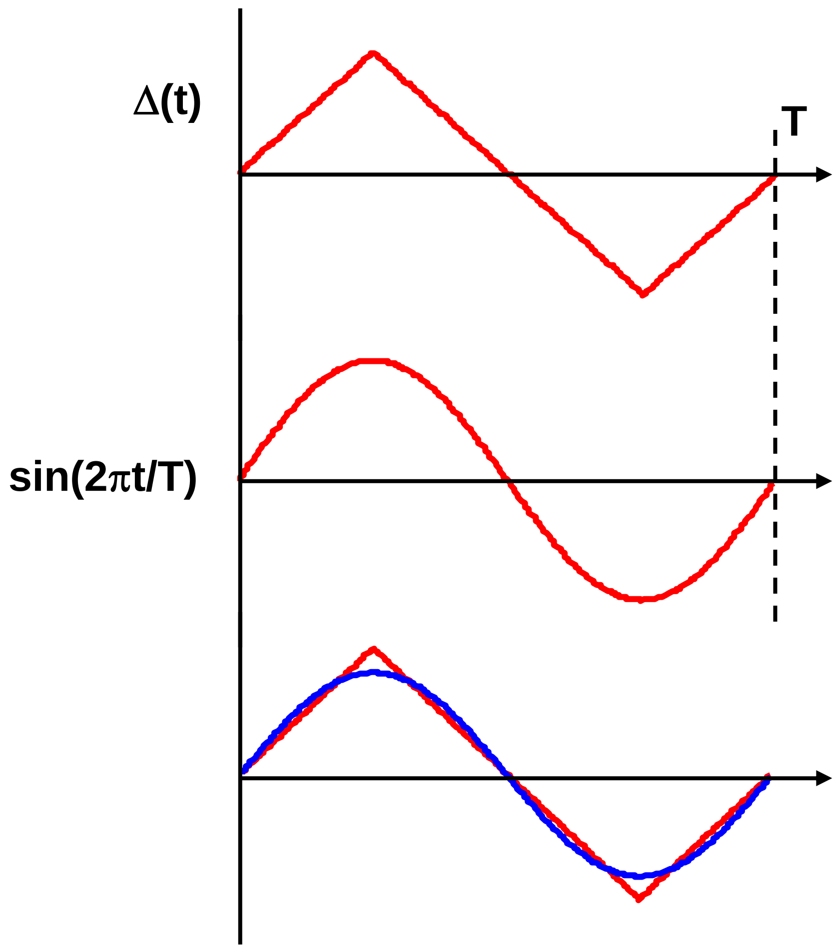


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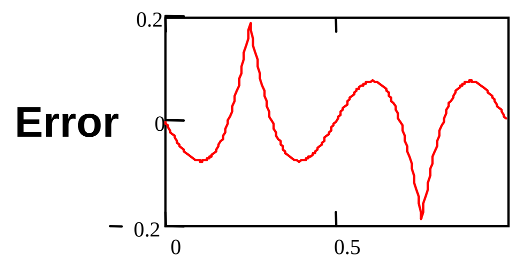

# Spectral Analysis of Arbitrary Signals

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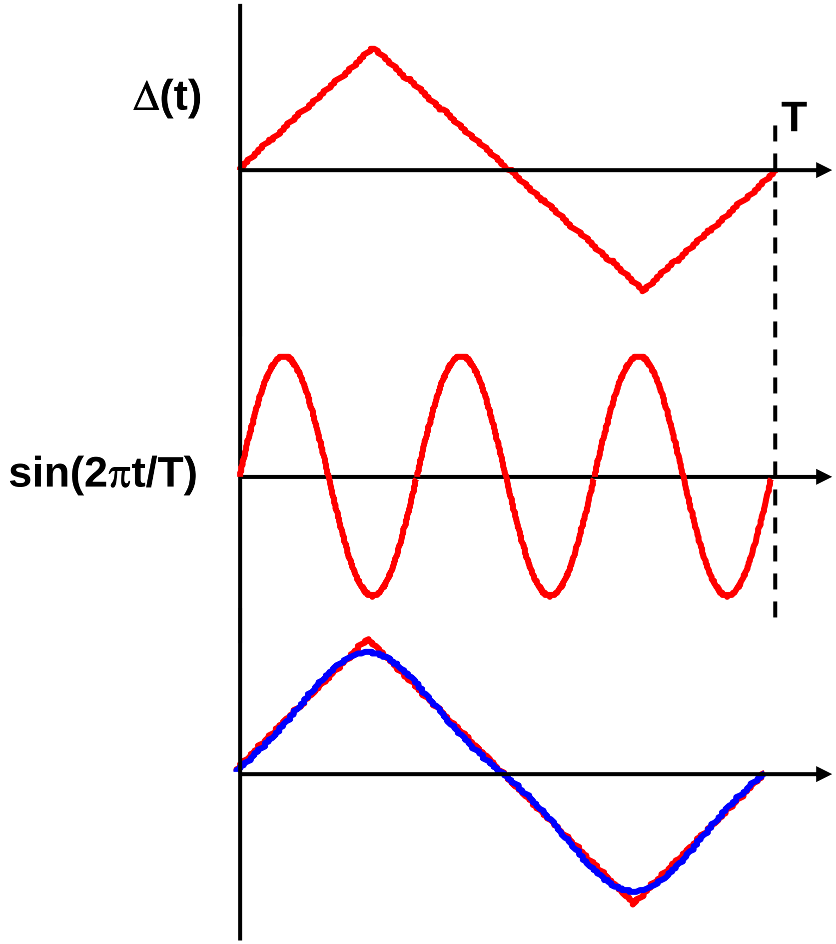
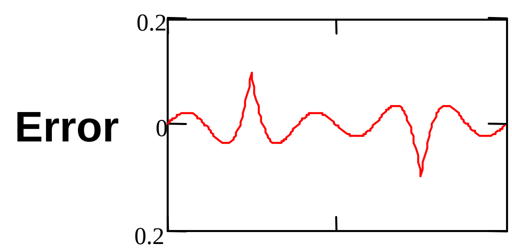



**How similar is this signal to a sinusoid with 3x the period?**

$$\frac{1}{\pi}\int_0^T \Delta(t)\sin\left(\frac{2\pi 3t}{T}\right)dt = -0.09$$



# Spectral Analysis of Arbitrary Signals

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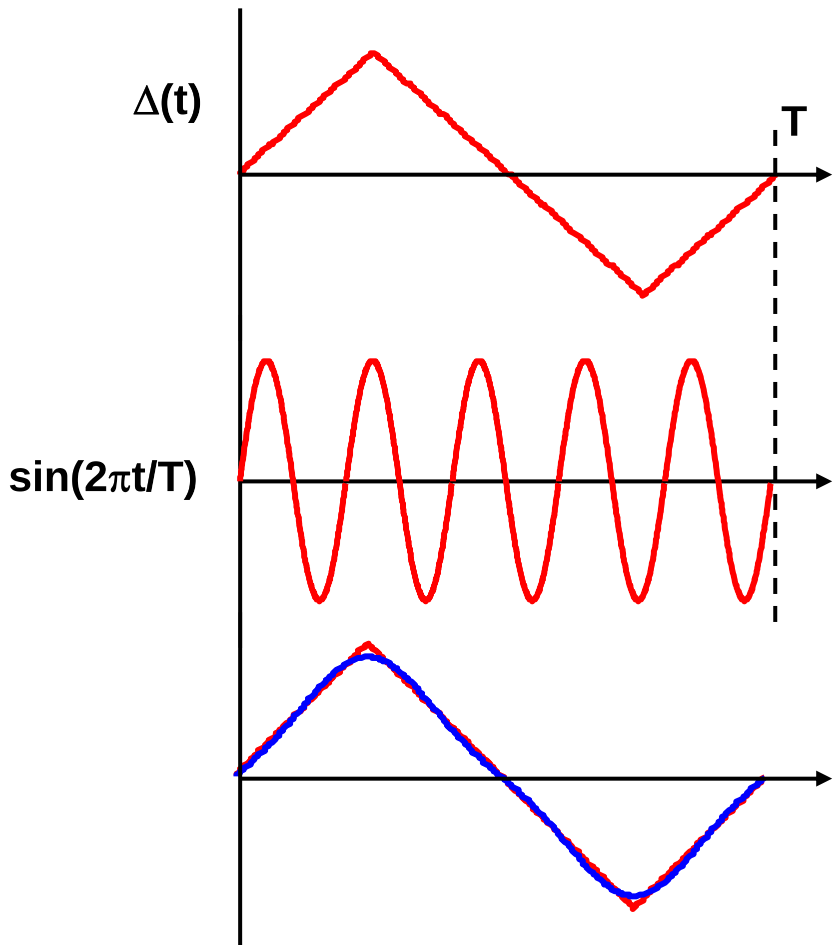
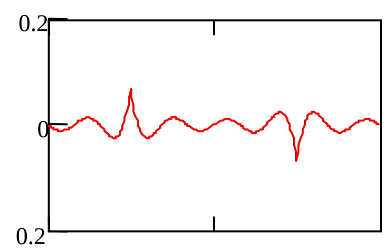



**How similar is this signal to a sinusoid with 5x the period?**

$$\frac{1}{\pi}\int_0^T \Delta(t)\sin\left(\frac{2\pi 5t}{T}\right)dt = 0.032$$

**Error**



# Spectral Analysis With Arbitrary Signals

- Any well-behaved periodic signal f(t) can be represented as

$$f(t) = a_0 + \left( \sum_{i=1}^{\infty} a_i \cos\left( i \frac{2\pi}{T} t \right) \right) + \left( \sum_{i=1}^{\infty} b_i \sin\left( i \frac{2\pi}{T} t \right) \right)$$

where

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left( k \frac{2\pi}{T} t \right) dt$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left( k \frac{2\pi}{T} t \right) dt$$

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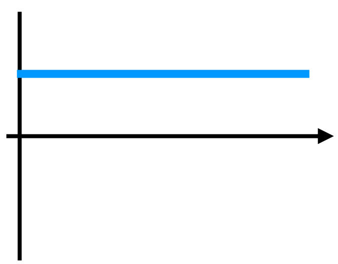
where

**DC component**

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left( k \frac{2\pi}{T} t \right) dt$$



$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left( k \frac{2\pi}{T} t \right) dt$$

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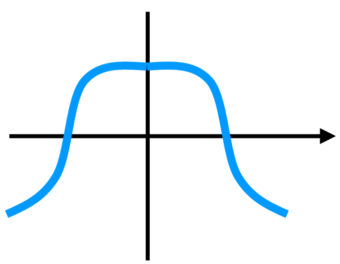
where

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

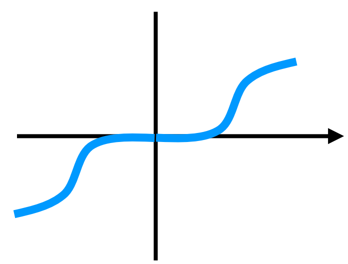
**Even function**

**Odd function**

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left( k \frac{2\pi}{T} t \right) dt$$

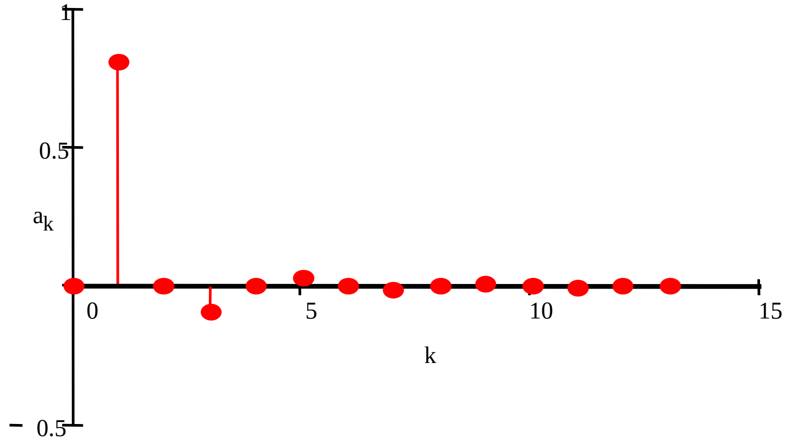


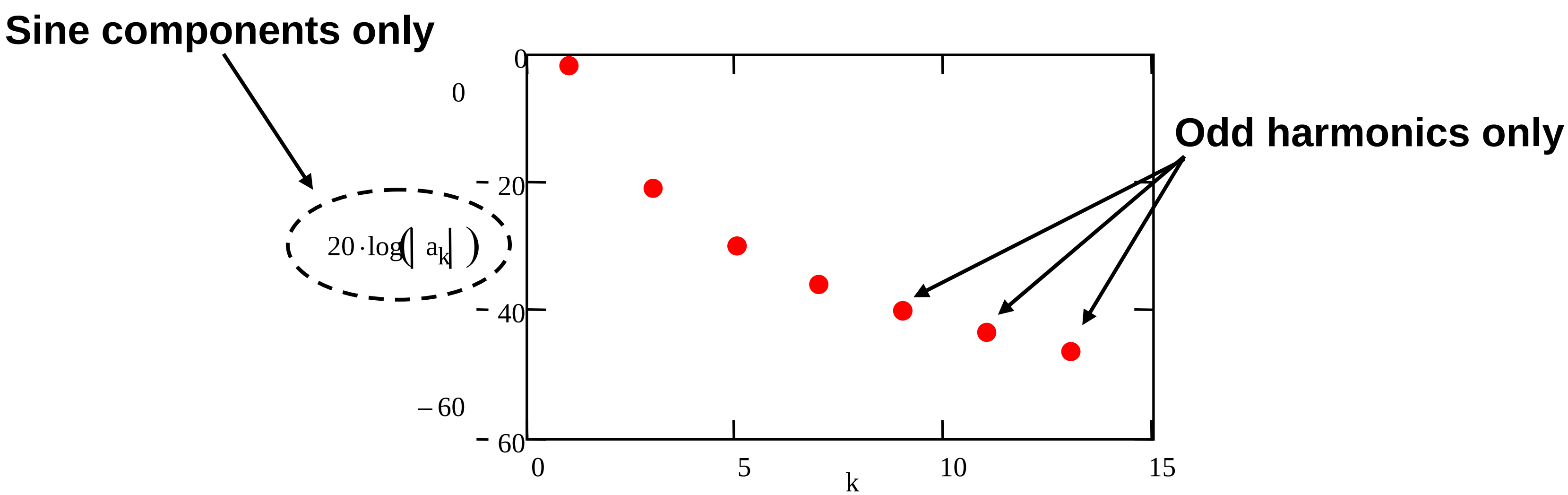
$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left( k \frac{2\pi}{T} t \right) dt$$



# Spectral Analysis With Arbitrary Signals

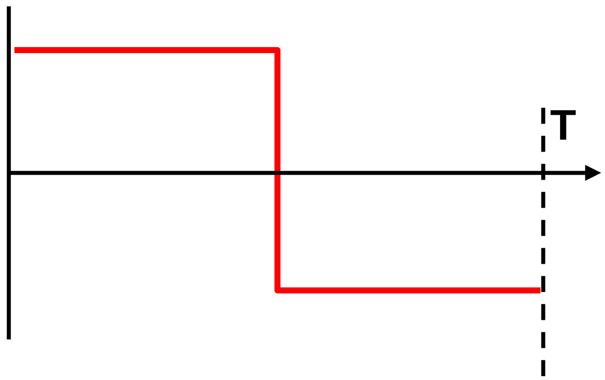
- Spectrum of triangular wave



# Spectrum Of A Few Common Signals

- Square wave



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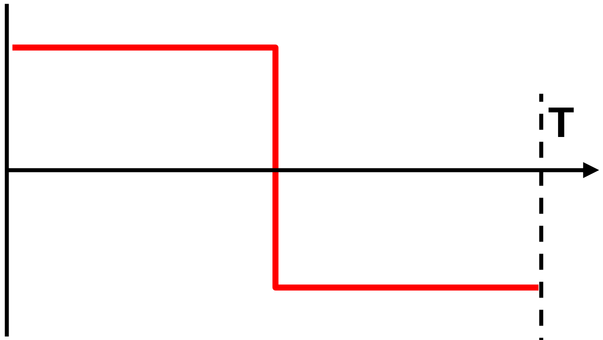
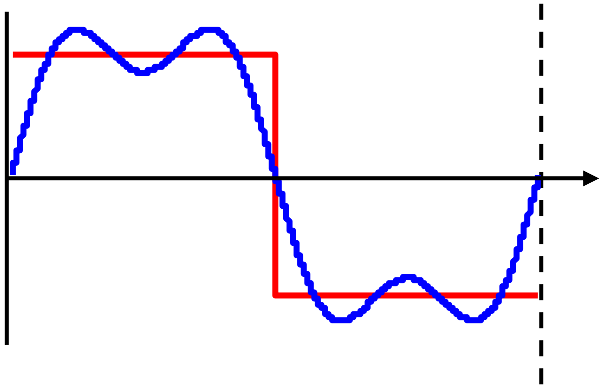
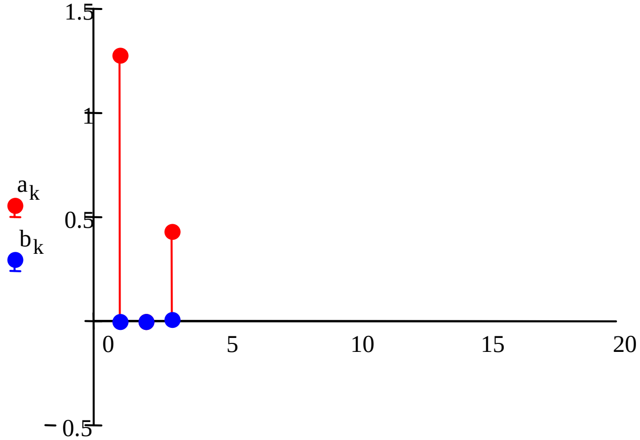


**N = 3**



# Spectrum Of A Few Common Signals

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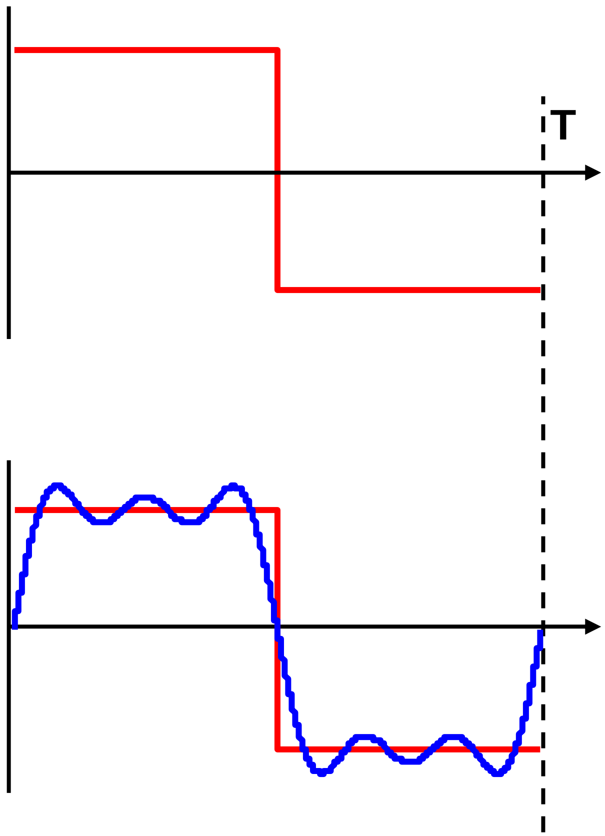
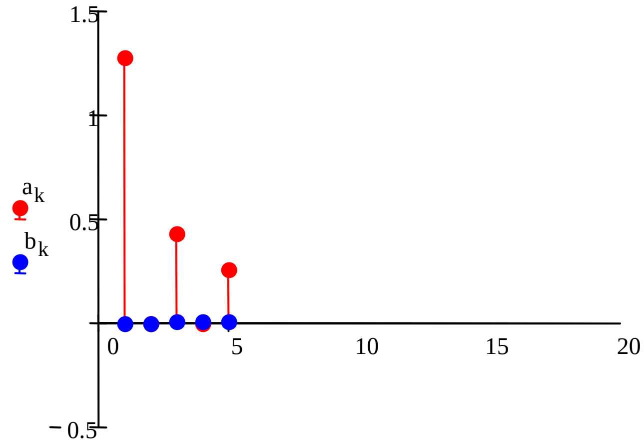
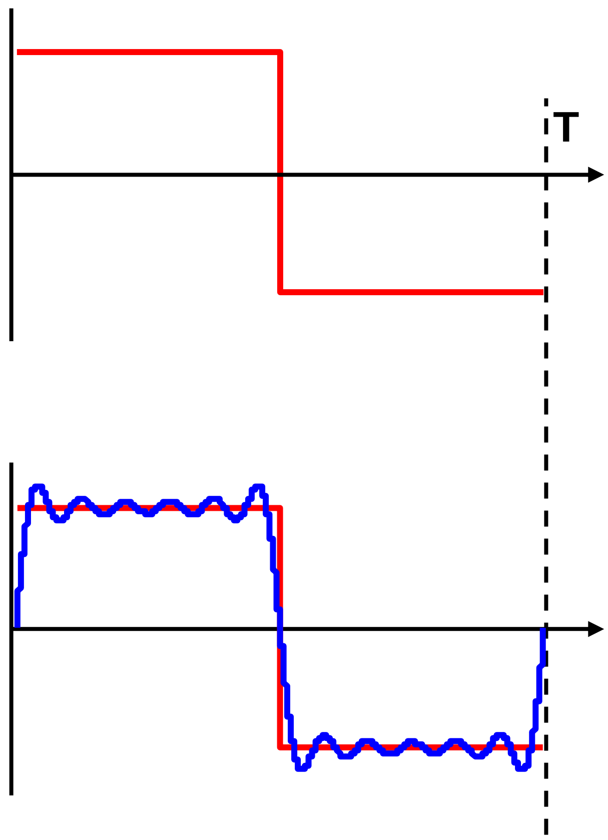



**N = 5**

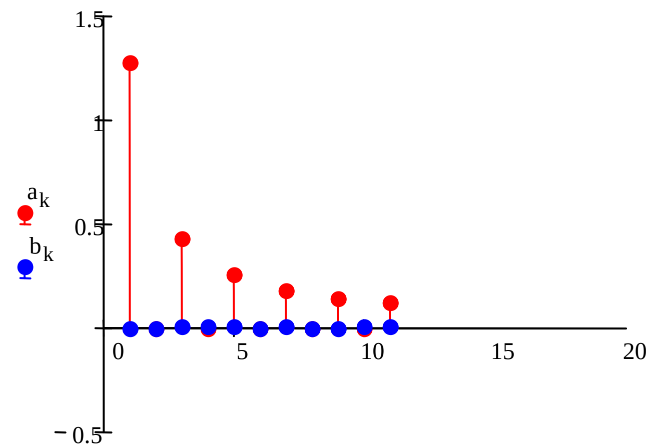


# Spectrum Of A Few Common Signals

- Square wave

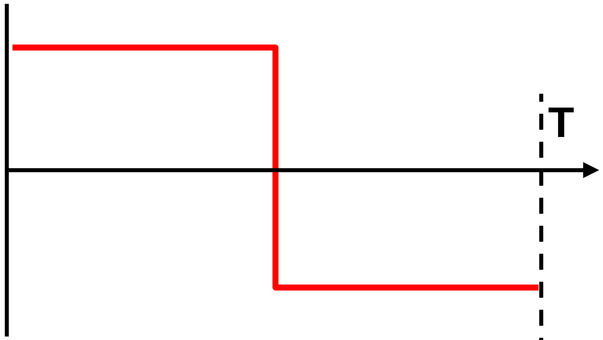


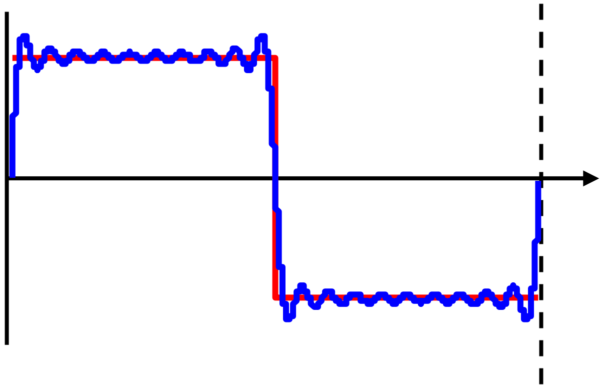

**N = 11**



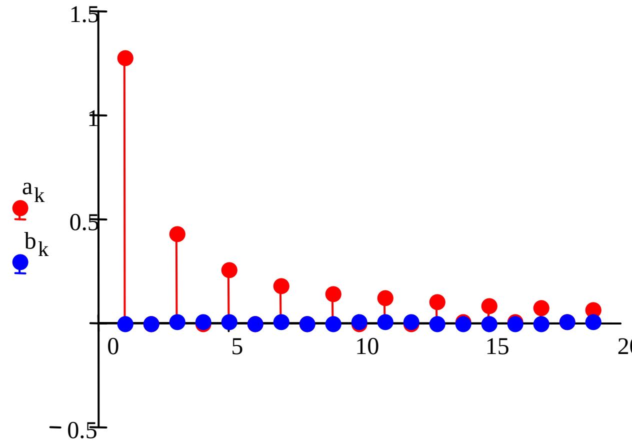
# Spectrum Of A Few Common Signals

- Square wave



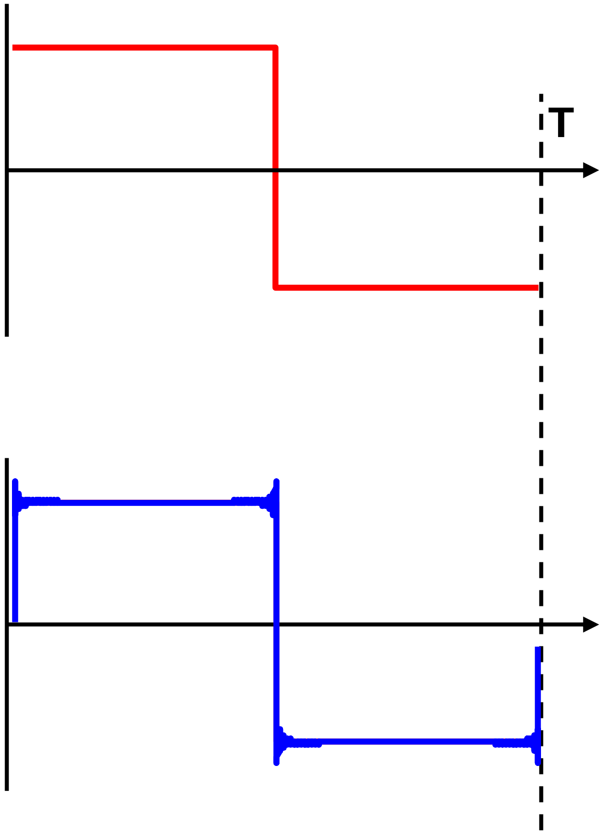



**N = 19**

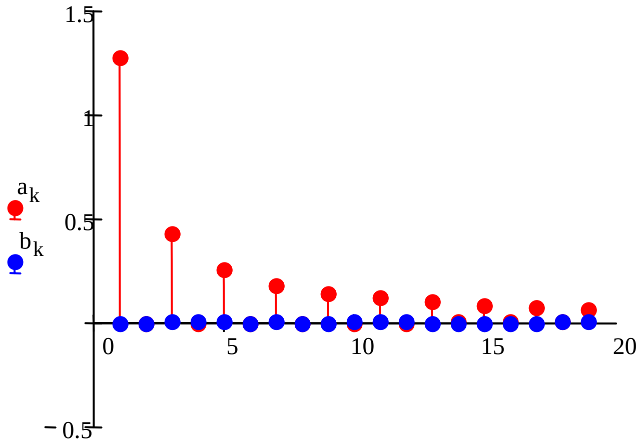


# Spectrum Of A Few Common Signals

- Square wave

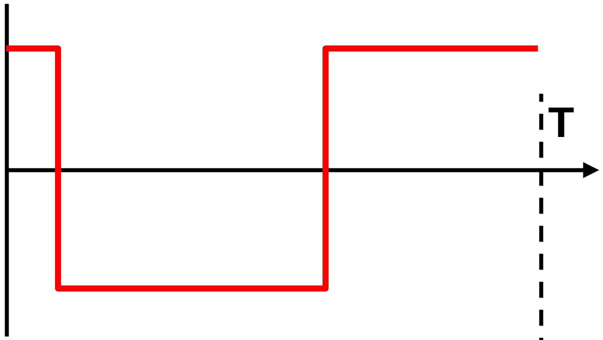


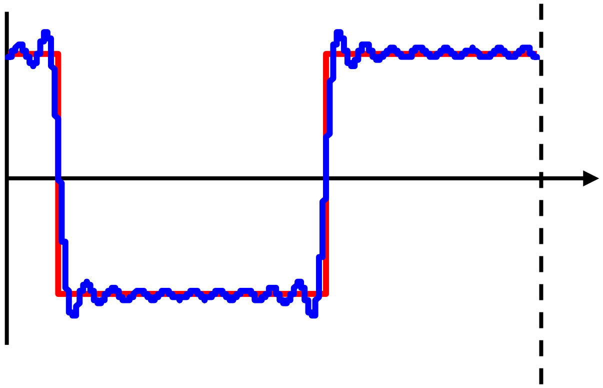

N = 200



# Spectrum Of A Few Common Signals

- Shifted square wave



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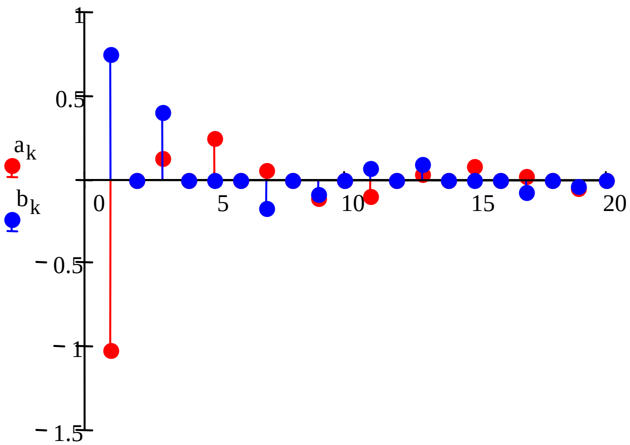
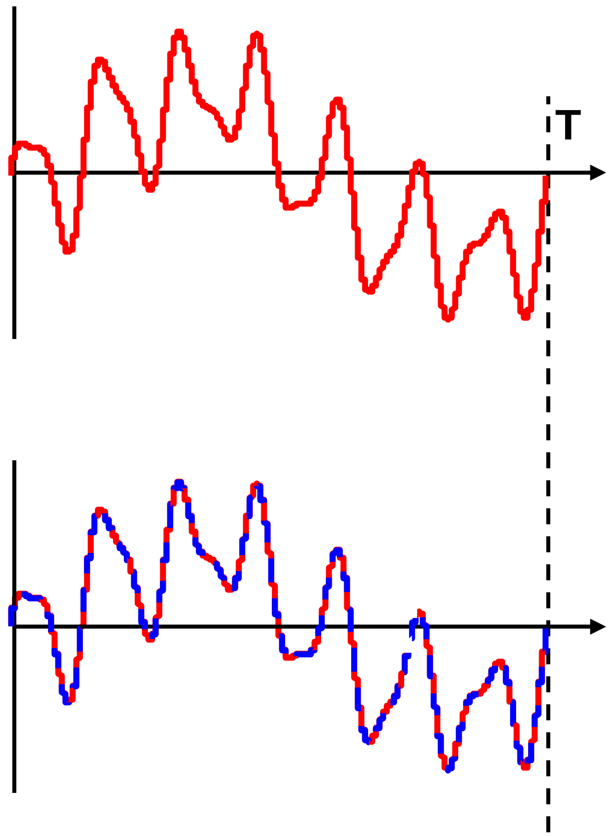


# Spectrum Of A Few Common Signals

- Arbitrary sum of sinusoids

$$f(x) = .9\sin(2\pi x) - .5\cos(2\pi x) + .1\sin(3\cdot 2\pi x) + .2\cos(5\cdot 2\pi x) + .8\sin(7\cdot 2\pi x) + .3\cos(13\cdot 2\pi x)$$

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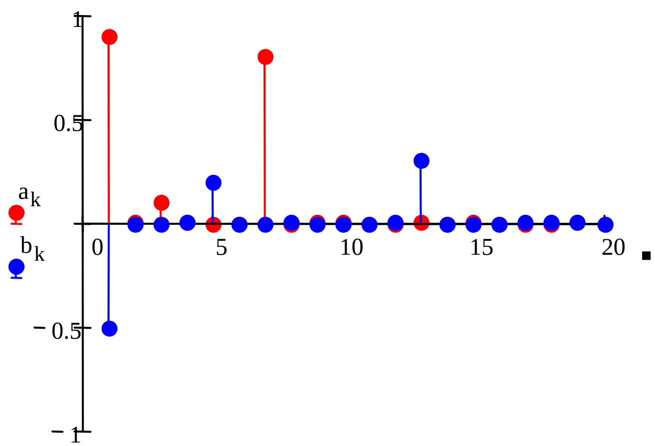
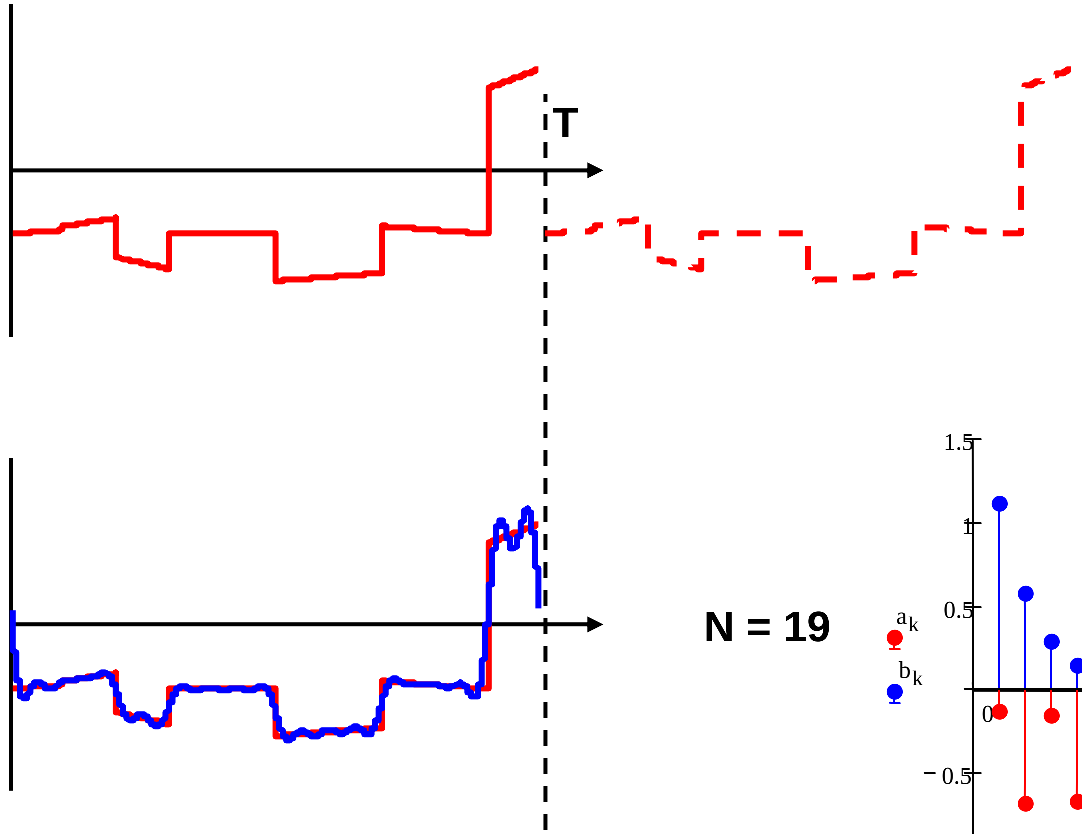


# Spectrum Of Non-Periodic Signals

- Treat the signal as though it is periodic

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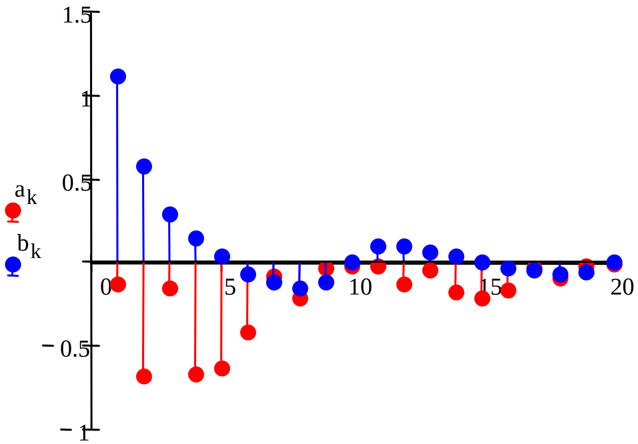

# Generalizing Fourier Series

$$f(t) = a_0 + \left( \sum_{i=1}^{\infty} a_i \cos\left( i \frac{2\pi}{T} t \right) \right) + \left( \sum_{i=1}^{\infty} b_i \sin\left( i \frac{2\pi}{T} t \right) \right)$$

**Fourier series**

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left( i \frac{2\pi}{T} t \right) dt$$

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# Generalizing Fourier Series

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**Euler's formula**

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

# Generalizing Fourier Series

**Fourier series**

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$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left( i \frac{2\pi}{T} t \right) dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi nt}{T}}$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j \frac{2\pi kt}{T}} dt$$

**Euler's formula**

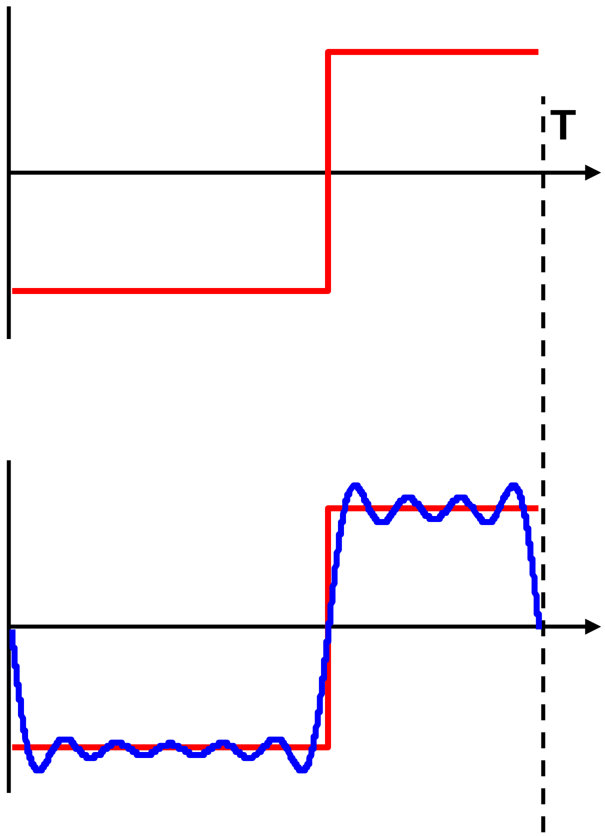
$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

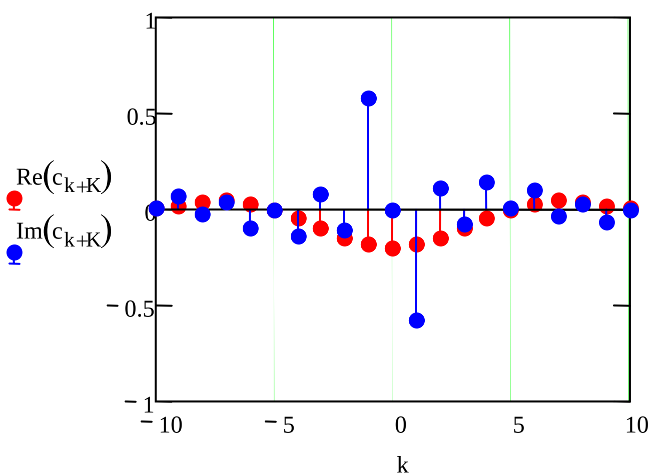
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

# Complex Spectrum Of A Signal

- Shifted square wave




**N = 19**



# Generalizing The Fourier Series

- Start with the complex Fourier Series

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$c_k=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi kt}{T}}dt$$

# Generalizing The Fourier Series

- Change variables

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

**Replace 2π/T with** $\omega_0$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi kt}{T}}dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt$$

# Generalizing The Fourier Series: The Fourier Transform

- Consider what happens when the analysis period is allowed to increase

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

**Replace 2π/T with $\omega_0$**

**Let $\omega_0$ go to 0**
**T becomes infinite**

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi kt}{T}}dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

# Applying The Fourier Transform To Sampled Signals

- Continuous samples of f(t), F(ω) are not available

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

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# Applying The Fourier Transform To Sampled Signals

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$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

$$f(n\Delta t) = \frac{1}{N}\sum_{k=0}^{N-1} F(k\Delta f)e^{j(2\pi k\Delta f)(n\Delta t)}$$

**Discrete time**

**Discrete frequency**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

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**N-point Discrete Fourier Transform (DFT)**

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**Discrete time**

**Note symmetry of $e^{jx}$**

**Not all $N^2$ factors need be calculated**

**Discrete frequency**

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

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~~**N-point Discrete Fourier Transform (DFT)**~~

**If $N=2^M$, (N x log(N)) operations needed for Fast Fourier Transform (FFT)**

# Next time

- Statistical Analysis of Experimental Data (read Ch. 6)